# Data-efficient Bayesian verification of parametric Markov Chains

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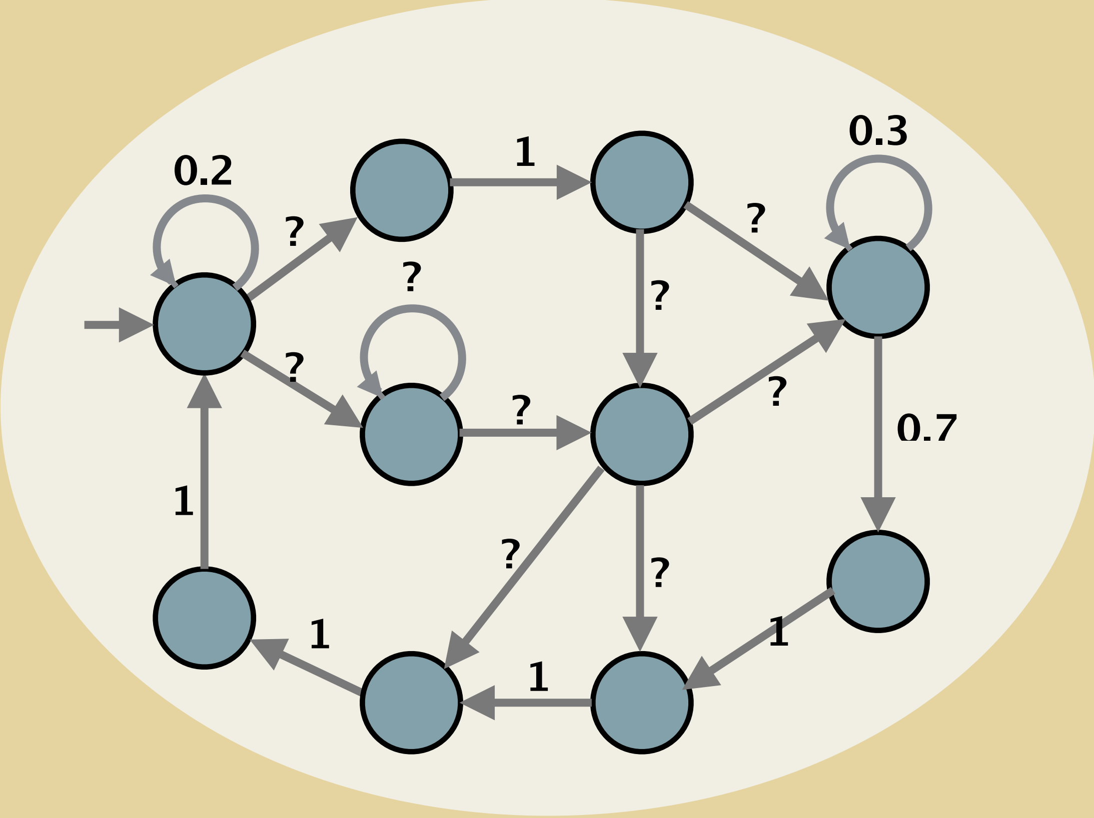
Model checking of systems with full models is well established ...

...but complete, accurate models are HARD to get.

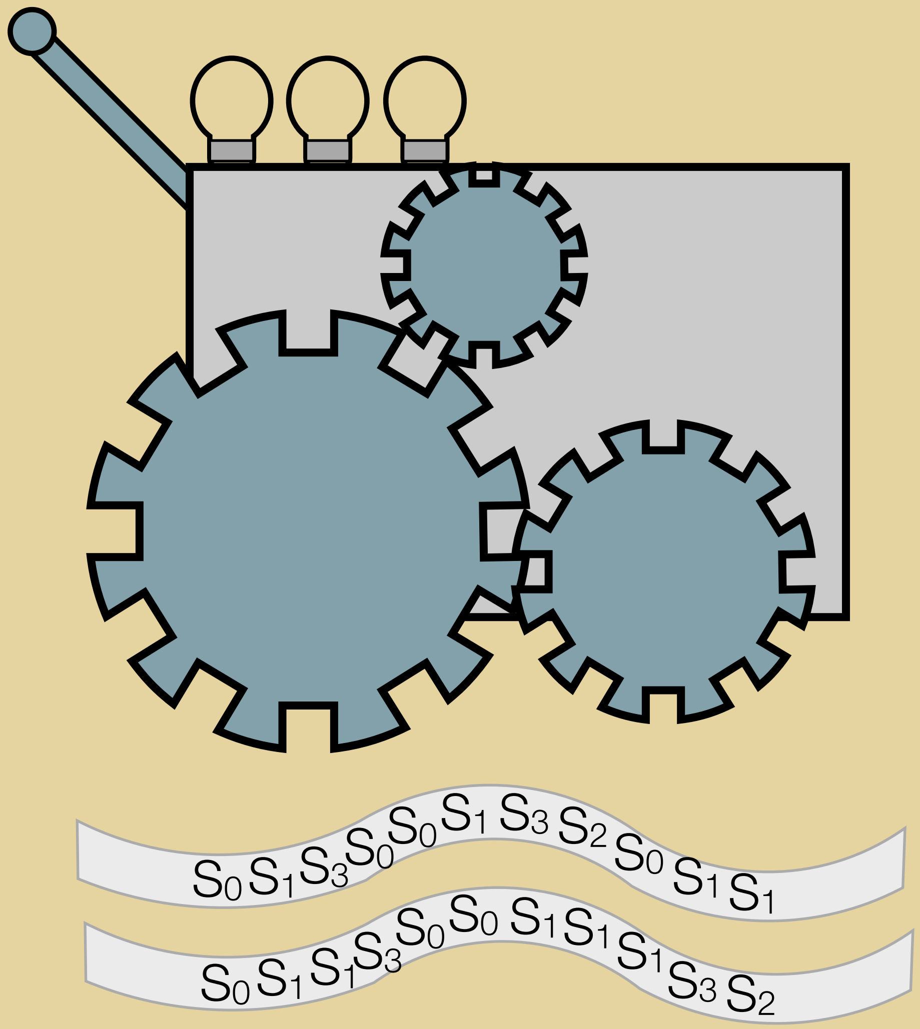


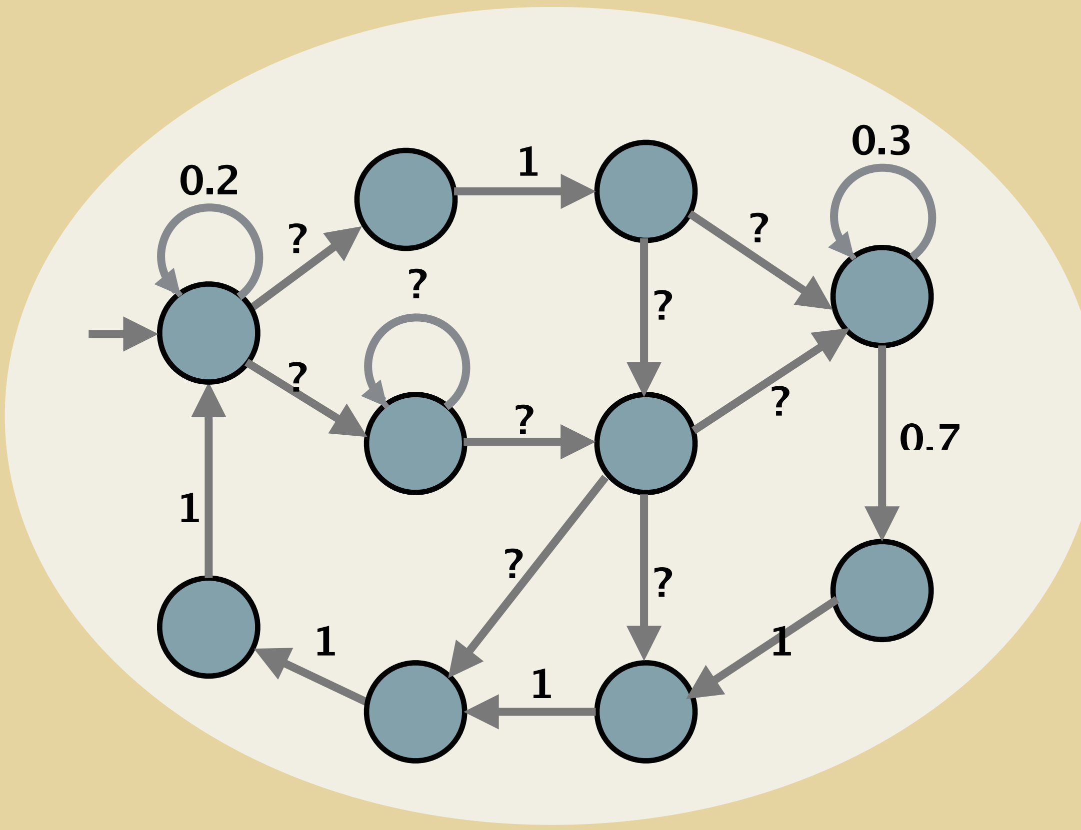

What can we do with a partial model?

Suppose we have a system,

we are given a partial model,

and a limited amount of system-generated data

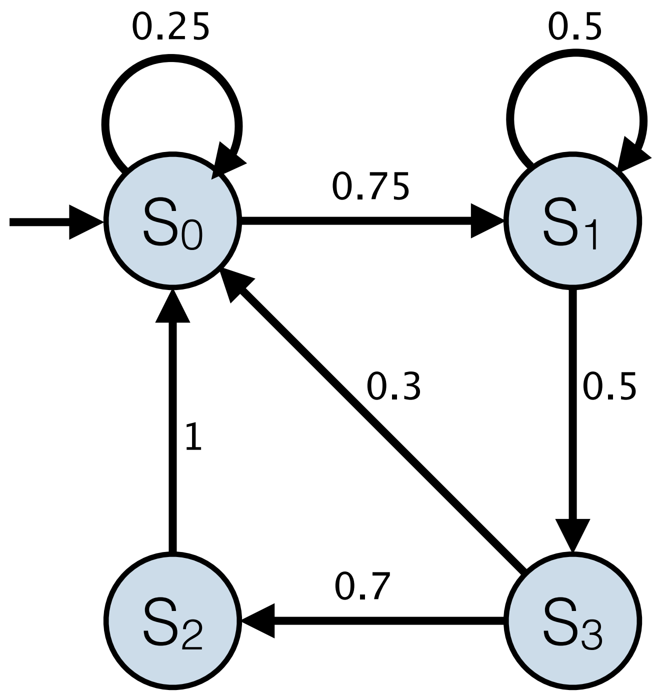



Can we check the system satisfies a PCTL property?

# Related work: “white-box” model checking

Explicit model checking: evaluate all possible paths in the **model**

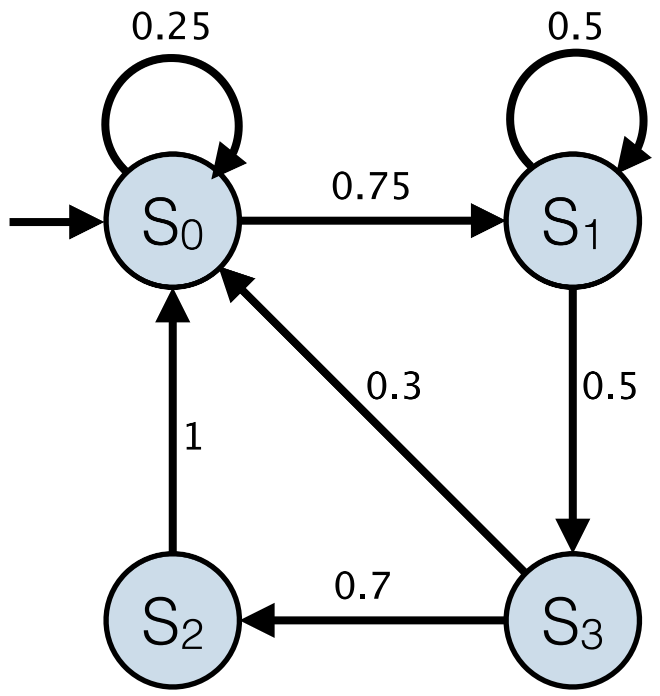



Proves that the **model** satisfies property

Relies on the model being correct and complete

# Related work: “white-box” model checking

Symbolic model checking: reason about all possible paths in the **model**

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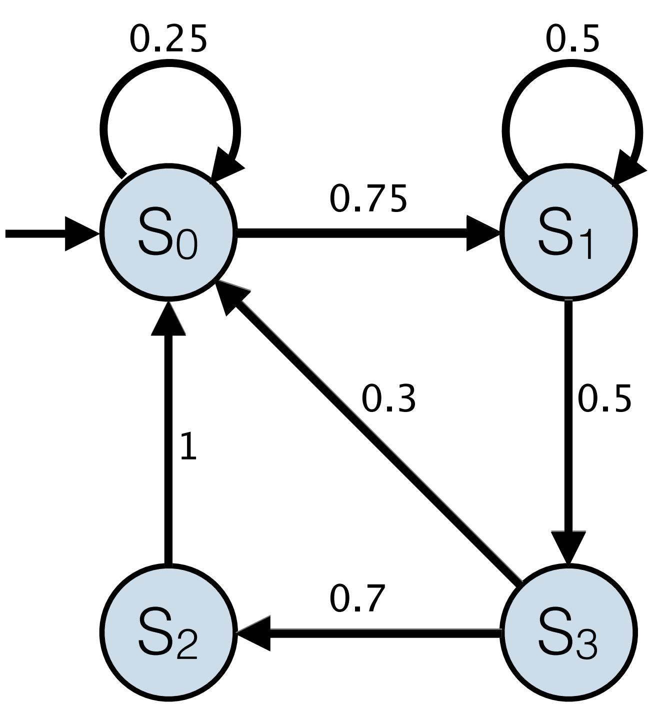
# Related work: “white-box” model checking

Statistical Model Checking (SMC): generate sample data from the **model**

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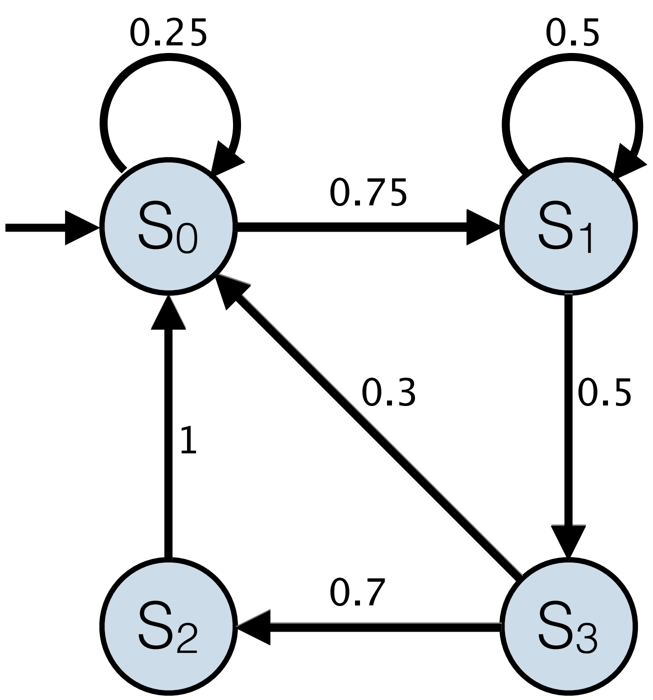
# Related work: "white-box" model checking

Statistical Model Checking (SMC): generate sample data from the **model**

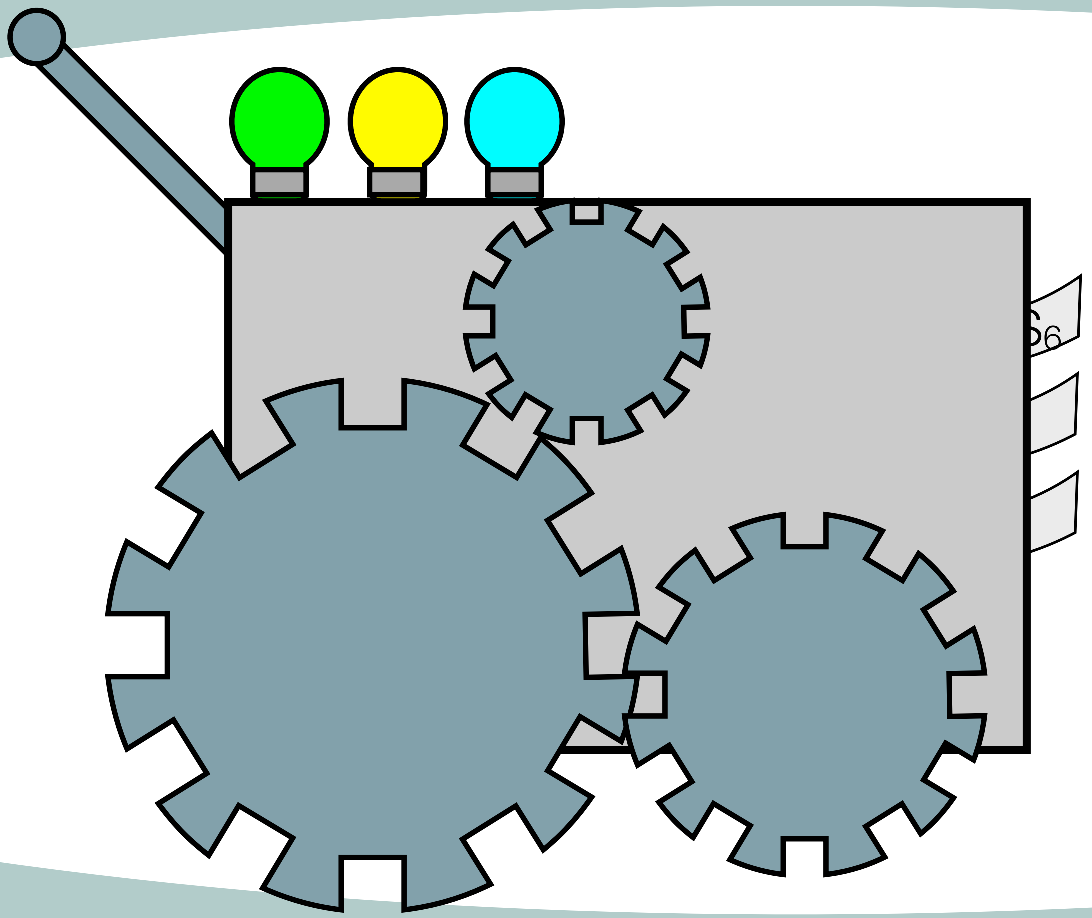



Gives probability that the **model** satisfies property, for BIG models

Relies on the model being correct and complete

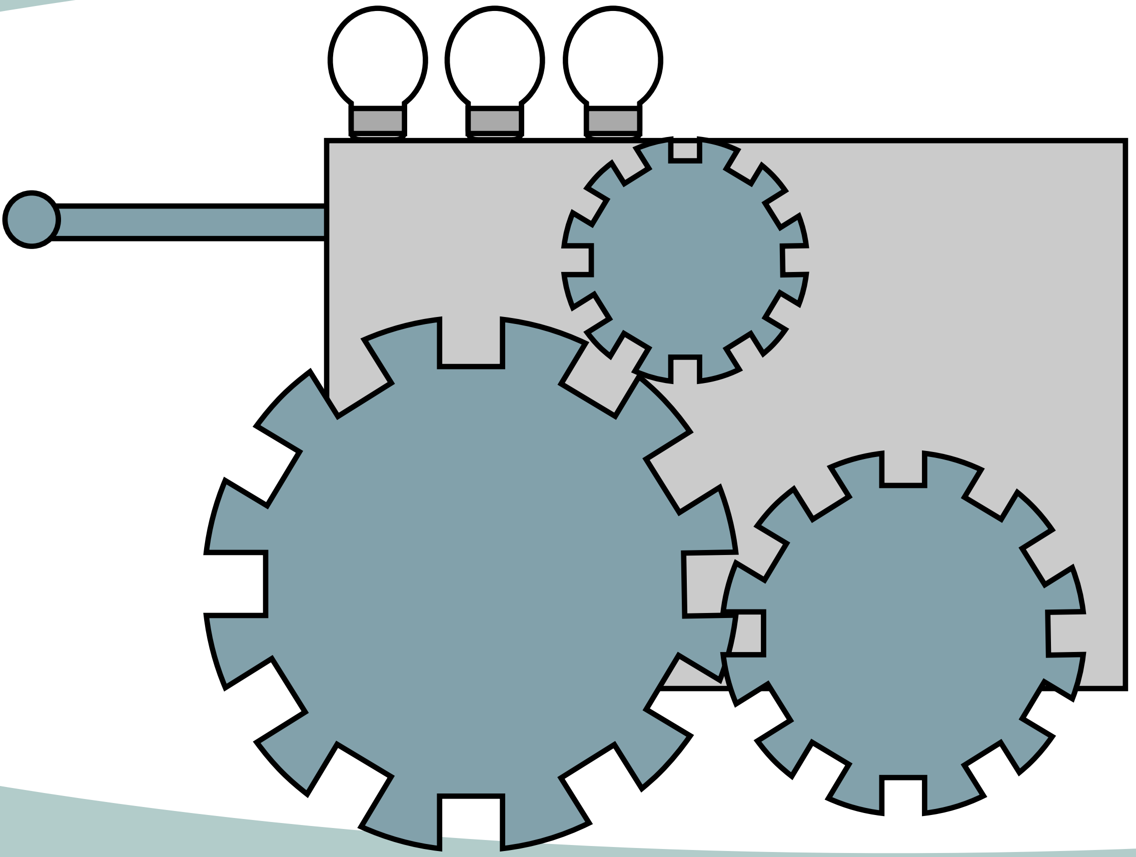
# Related work: "black-box" model checking

Statistical Model Checking (SMC): collect sample data from the **system**



# Related work: “black-box” model checking

Statistical Model Checking (SMC): collect sample data from the **system**



Gives probability that **system** satisfies property.

Needs a lot of data.

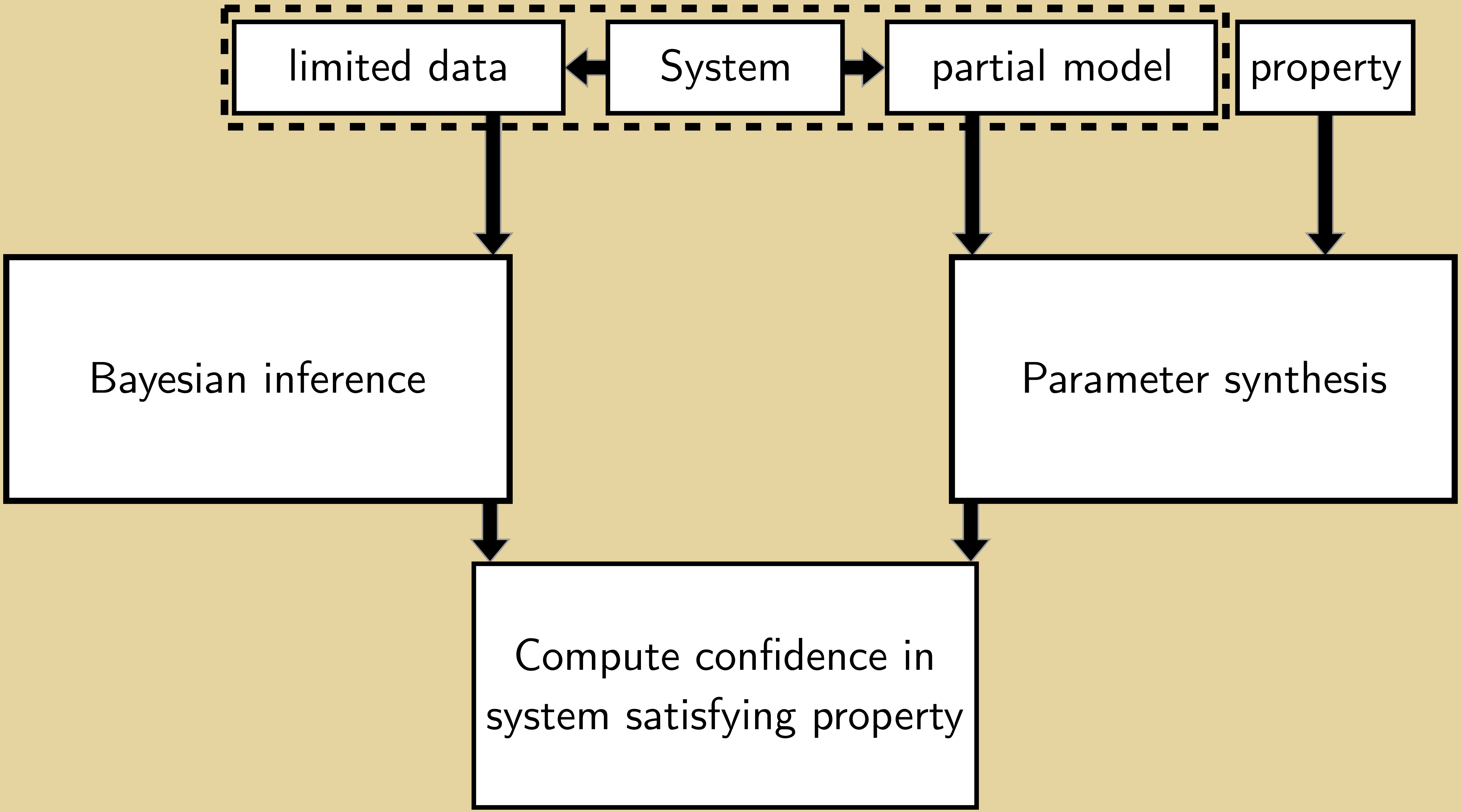
# Our approach:

Consider a scenario with limited data, so we can't use "black-box" SMC,

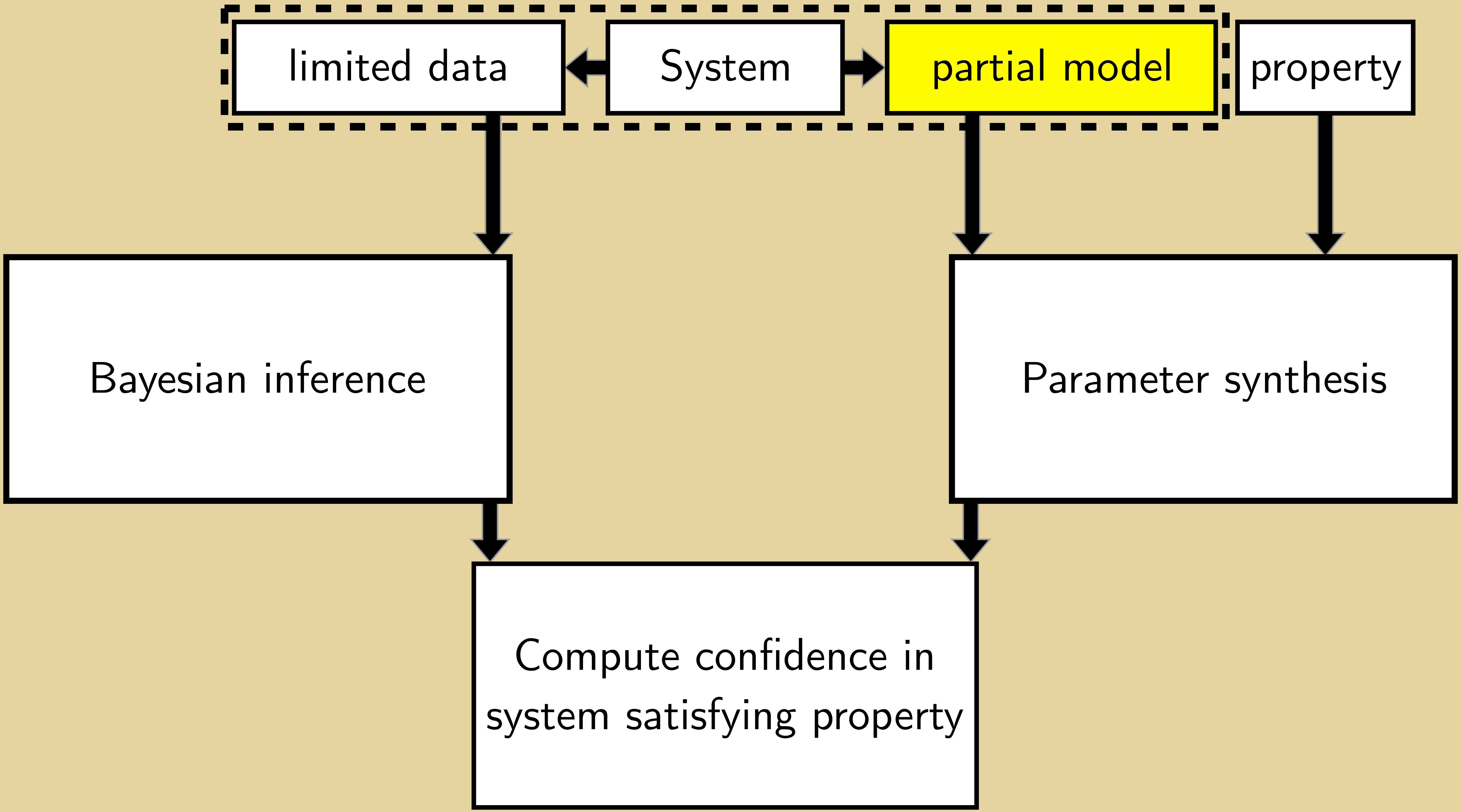
and only a partial model, so we can't use "white-box" model checking.

We combine parameter synthesis + data-based learning to compute the confidence the system satisfies a property.

# Overview


limited data
System
partial model
property
Bayesian inference
Parameter synthesis
Compute confidence in system satisfying property

# Overview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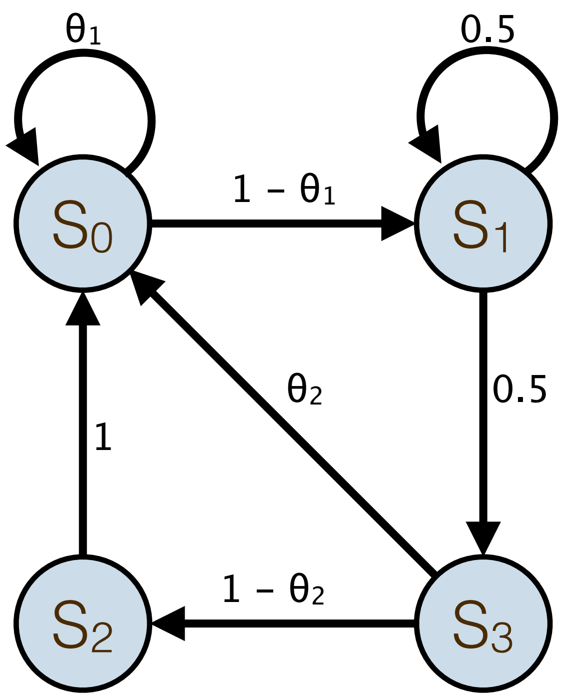
# Parametric Markov chains




basic pMC

*initial state*

*atomic propositions*

S = *finite set of states*

*labelling function*

$$\mathbf{M}_{\Theta} = (S, \mathbb{T}_{\theta}, \iota_{init}, \mathrm{AP}, L, \Theta)$$

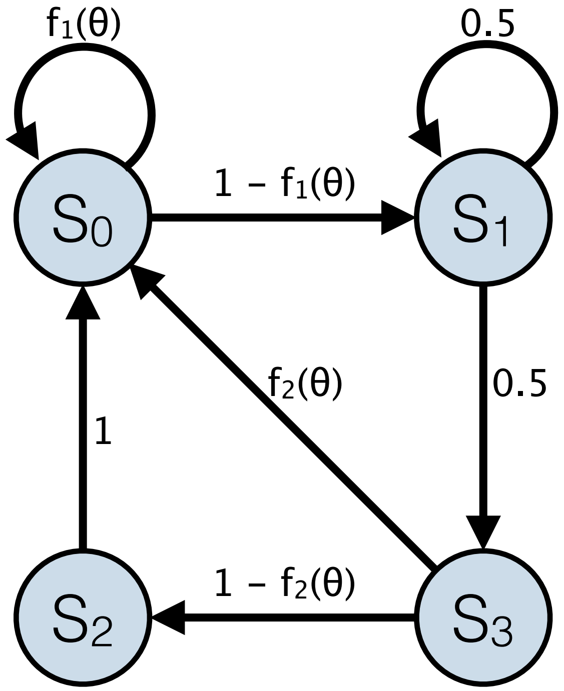
$\mathbb{T}_{\theta}$ = *parameterised transition function*

$\theta$ = *parameter vector*

$\Theta$ = *set of all possible valuations of* $\theta$

Basic pMC - transition probabilities are known constants or single parameters

# Parametric Markov chains




linear pMC

$$\mathbf{M}_{\Theta} = (S, \mathbb{T}_{\theta}, \iota_{init}, \mathrm{AP}, L, \Theta)$$

S = *finite set of states*

*initial state*

*atomic propositions*

*labelling function*

$\mathbb{T}_{\theta}$ = *parameterised transition function*

$\theta$ = *parameter vector*

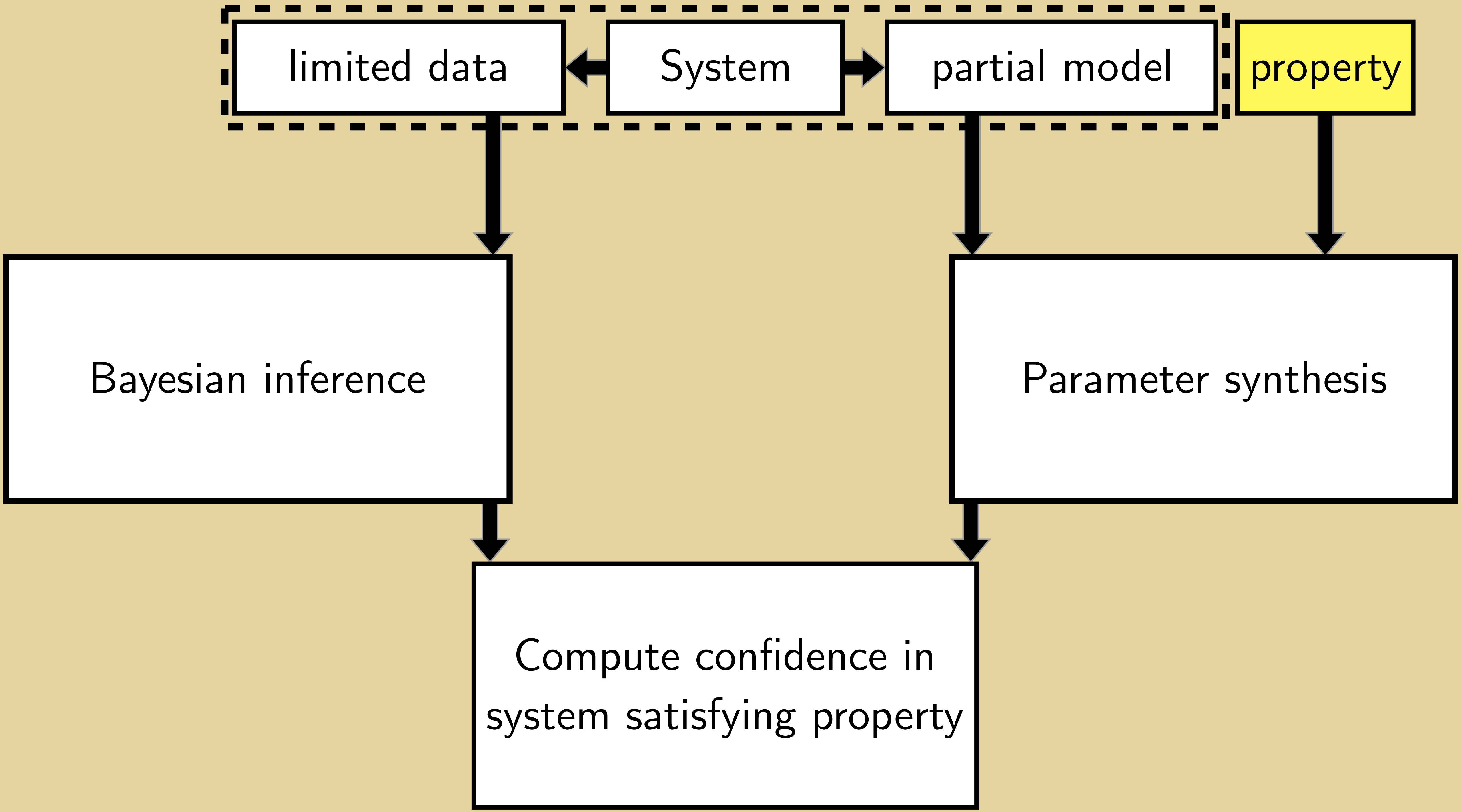
$\Theta$ = *set of all possible valuations of* $\theta$

Linear pMC - transition probabilities are linear functions of parameters

$$f_l(\theta) = k_0 + k_1\theta_1 + k_2\theta_2 + \ldots + k_n\theta_n$$

$$k_i \in [0,1] \qquad \sum k_i \leq 1$$

# Overview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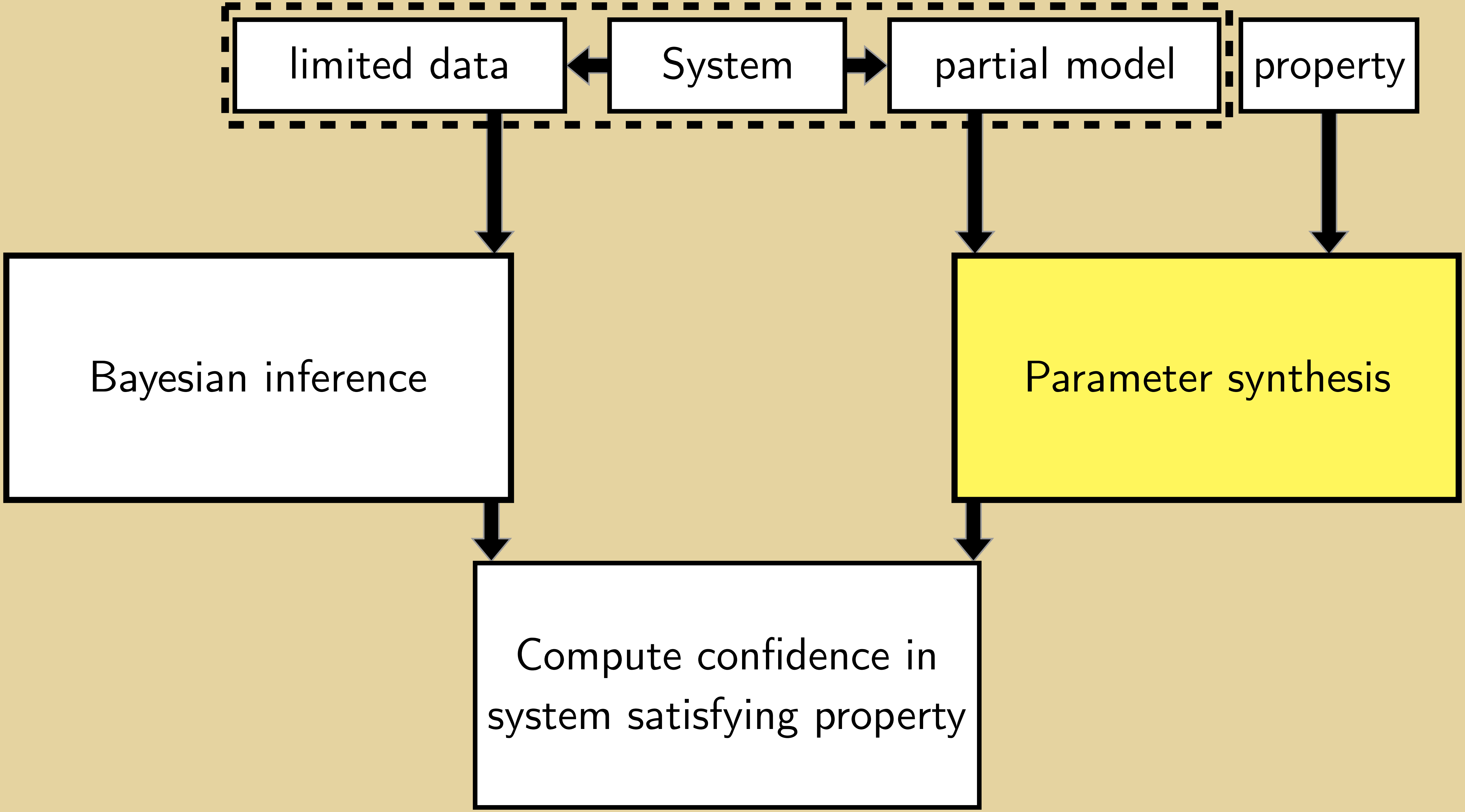
Parameter synthesis
Compute confidence in
system satisfying property

# PCTL properties

We are able to consider any property that is compatible with the PRISM parameter synthesis tool. We focus on non-nested PCTL

$$\phi = P_{>0.5}[G \neg s_2]$$

# Overview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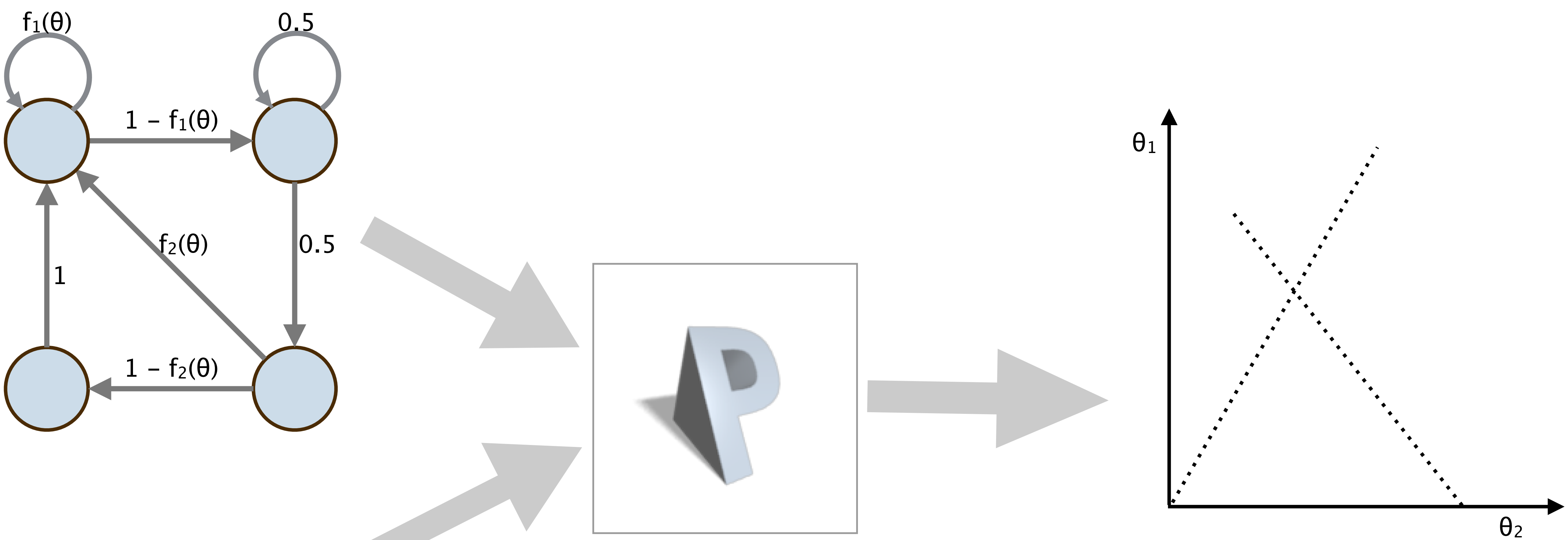
# Parameter Synthesis

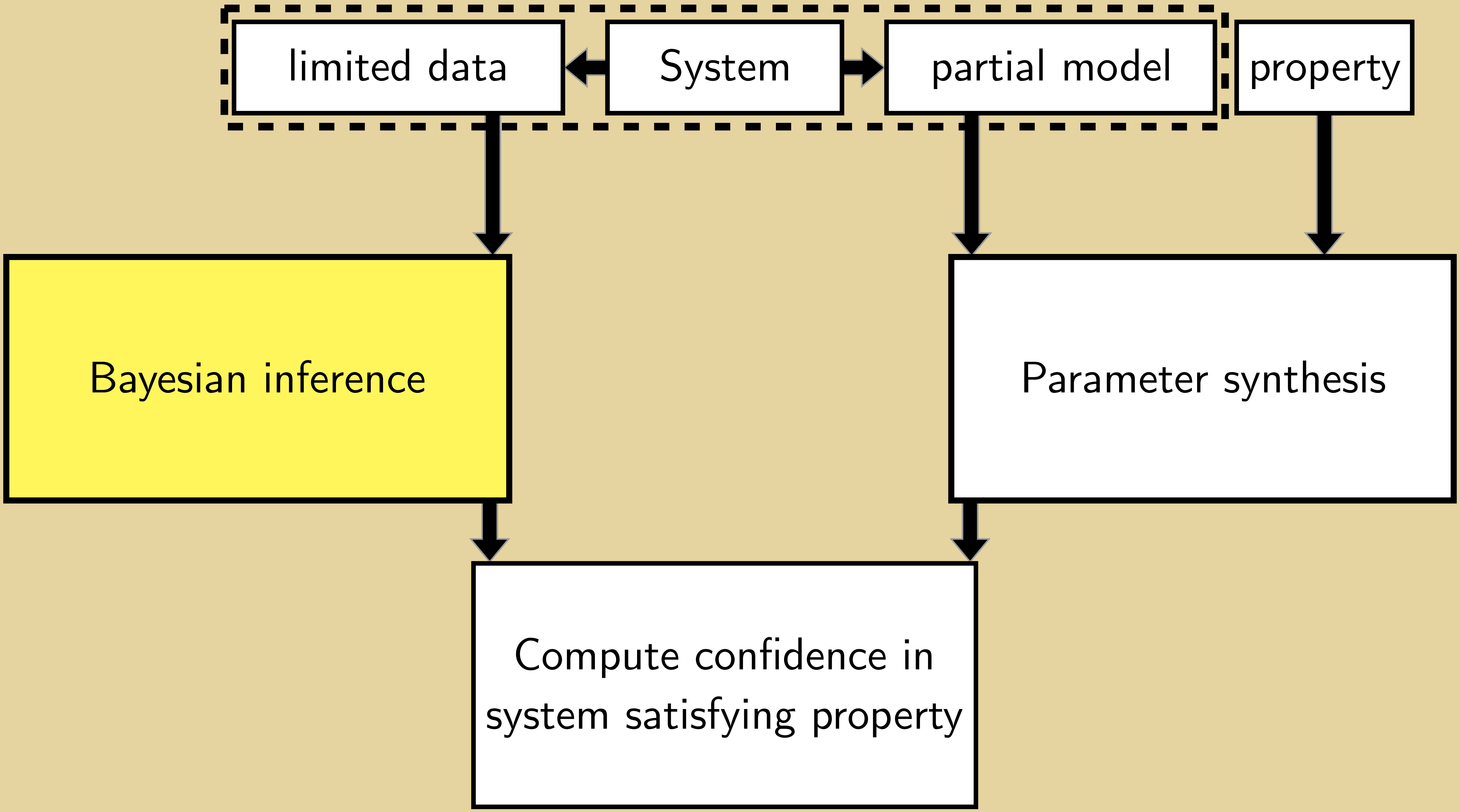
We use PRISM to synthesise the feasible set of parameters, for which the model satisfies the property




$\phi = P_{>0.5}[G \neg s_2]$

$$\Theta_\phi = \{\theta \in \Theta : \mathbf{M}(\theta) \models \phi\}$$

# Overview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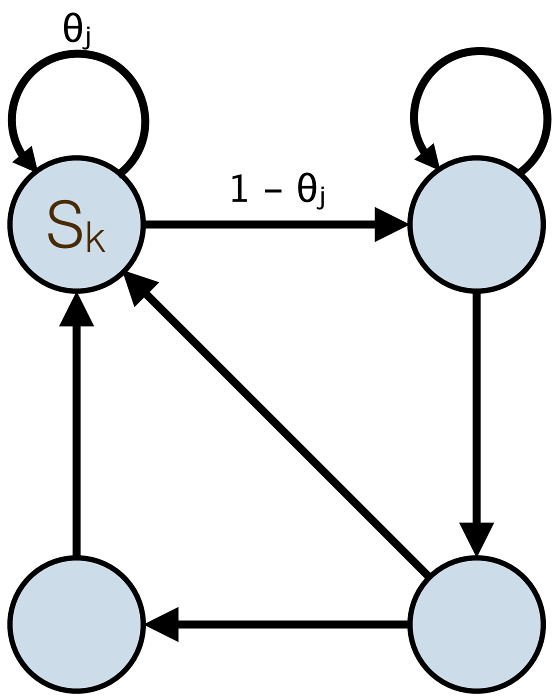
# Bayesian Inference: basic pMC

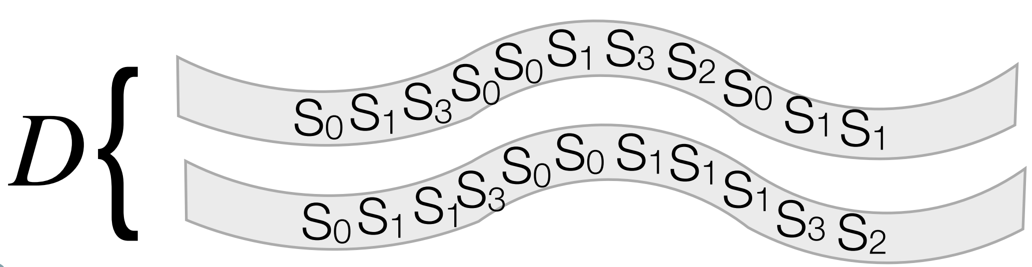


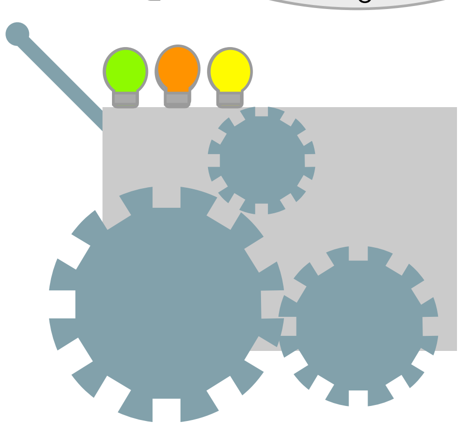

*observed data* *prior*

$$p(\theta_j \mid D) = \frac{\mathbb{P}(D \mid \theta_j)p(\theta_j)}{\mathbb{P}(D)}$$

$$= \frac{p(\theta_j)\boxed{\prod_{s' \in S} \mathbb{T}_\theta(s_k, s')^{D_{s_k}^{s'}}}}{\mathbb{P}(D_{s_k})}$$

binomial distribution

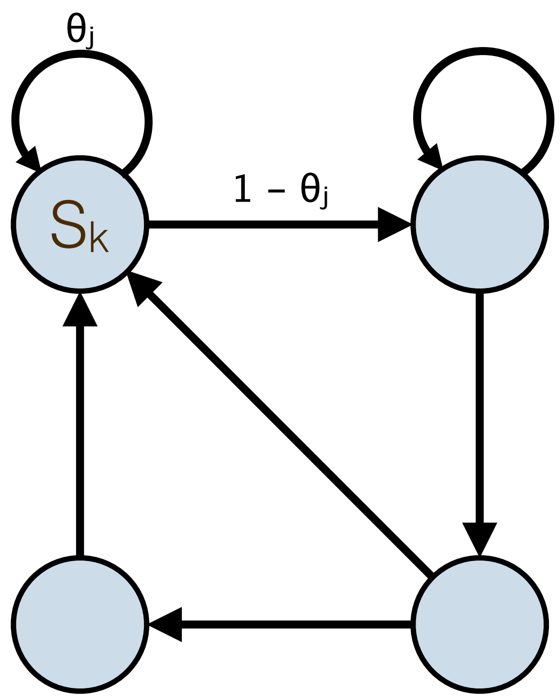


Conjugate prior = Dirichlet

$$\mathrm{Dir}(\theta_j \mid \alpha) = \tfrac{1}{B(\alpha)}\theta_j^{\alpha_1 - 1}(1-\theta_j)^{\alpha_2 - 1}$$

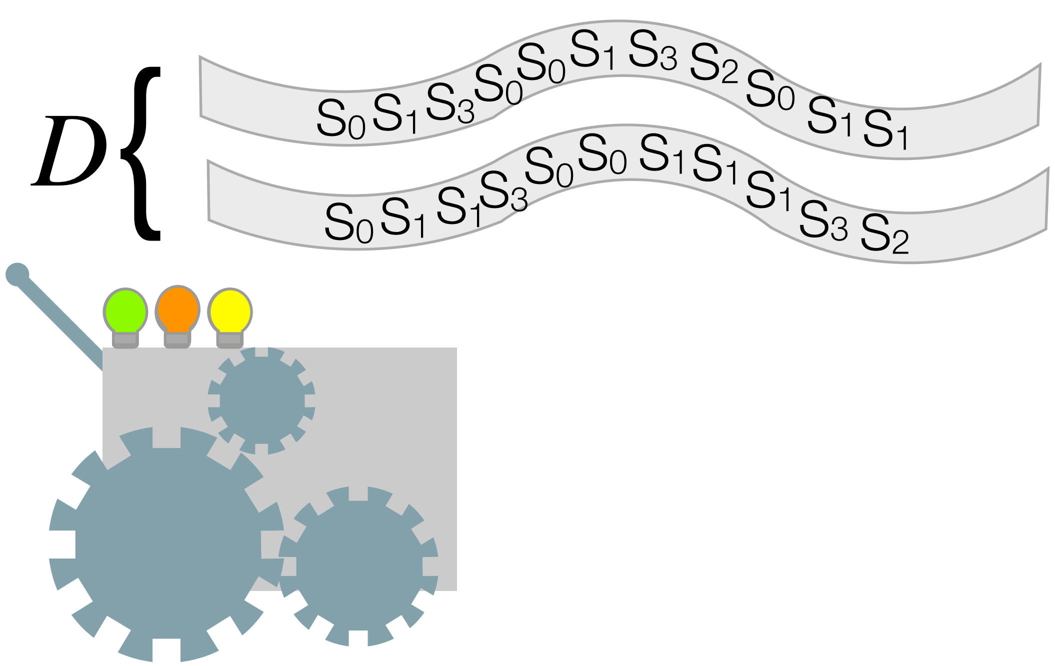
# Bayesian Inference: basic pMC




*observed data* *prior*

$$p(\theta_j \mid D) = \frac{\mathbb{P}(D \mid \theta_j)p(\theta_j)}{\mathbb{P}(D)}$$

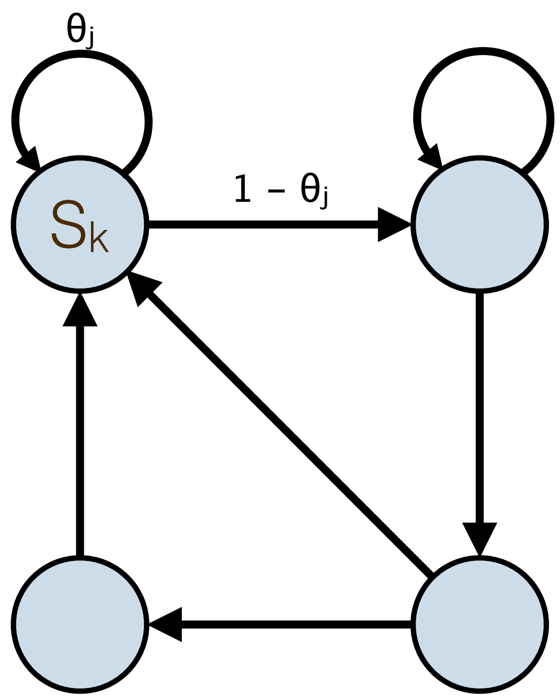
$$p(\theta_j \mid D) \propto \theta_j^{\alpha_1 - 1}(1-\theta_j)^{\alpha_2 - 1}\theta_j^{D_{s_k}^{s_1}}(1-\theta_j)^{D_{s_k}^{s_2}}$$



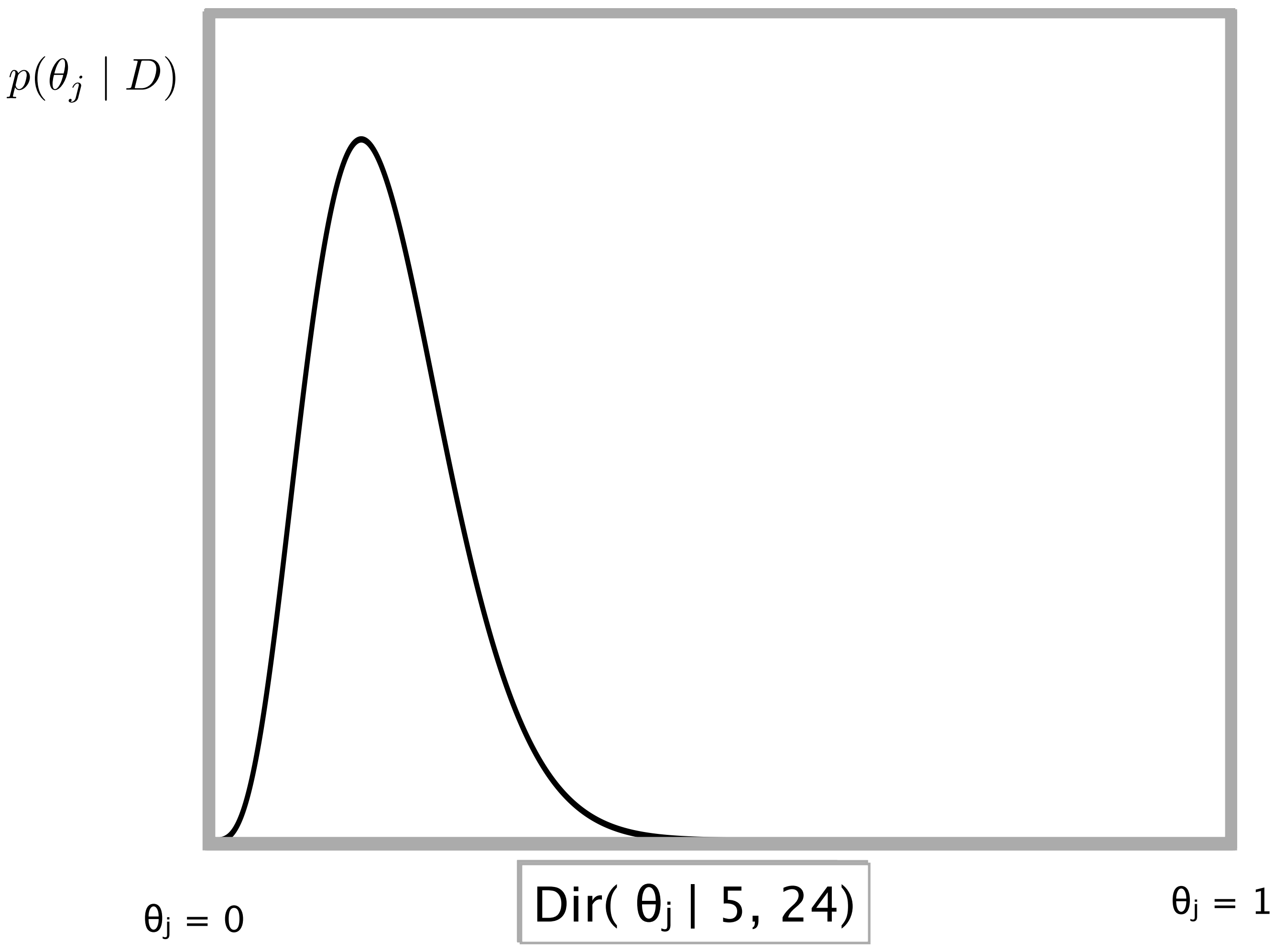

Hence, updating posterior = adding transition count to Dirichlet hyper-parameters

$$p(\theta \mid D) = \prod_{s_i} \mathrm{Dir}(\theta_{s_i} \mid D_{s_i} + \alpha)$$

# Bayesian Inference: basic pMC




| | |
|---|---|
| COUNT [1 – θj] | 23 |
| COUNT [θj] | 4 |

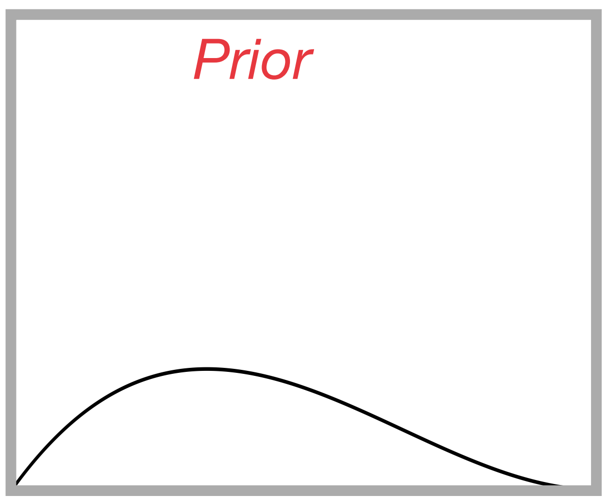


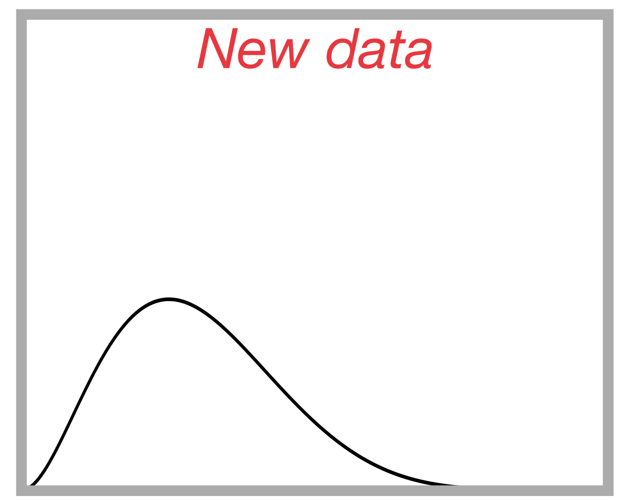
# Combining posterior distributions

Note we can combine posterior distributions from multiple identically parameterised transitions by summing the hyper-parameters

COUNT [1 – $\theta_j$] 2

COUNT [$\theta_j$] 1




COUNT [1 – $\theta_j$] 8

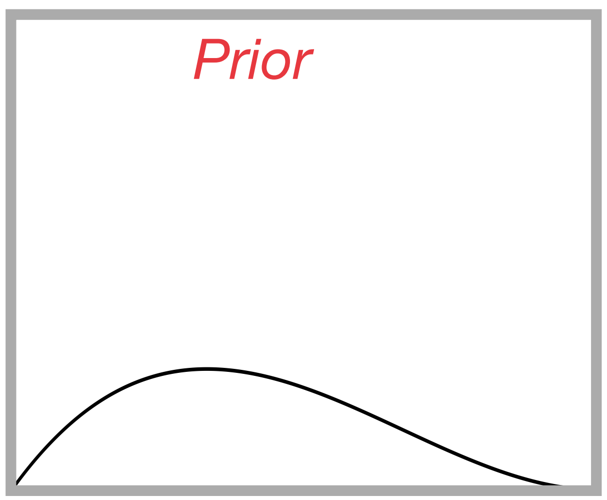
COUNT [$\theta_j$] 3

$$p(\theta_j) = \mathrm{Dir}(D_{s_1} + D_{s_2} + \alpha_{s_1})$$

# Combining posterior distributions

Note we can combine posterior distributions from multiple identically parameterised transitions by summing the hyper-parameters

$$\frac{\text{COUNT}\,[1 - \theta_j] \quad 2}{\text{COUNT}\,[\theta_j] \quad 1}$$

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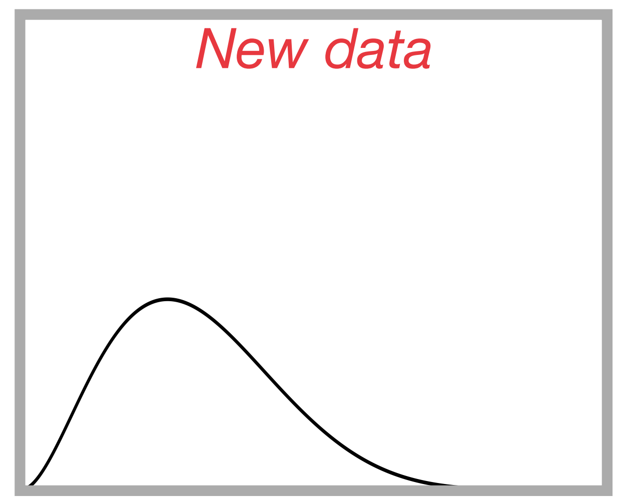
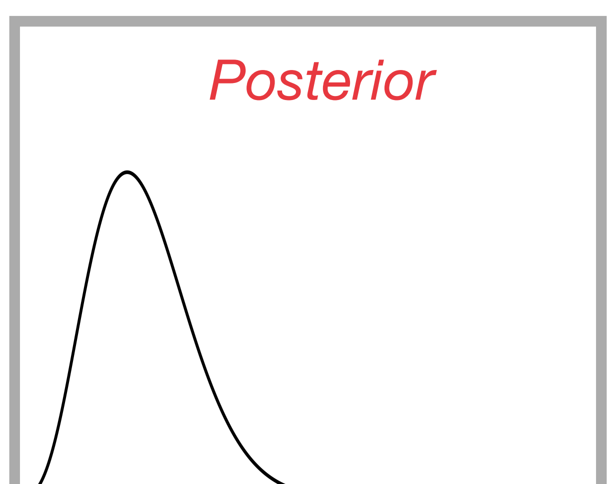


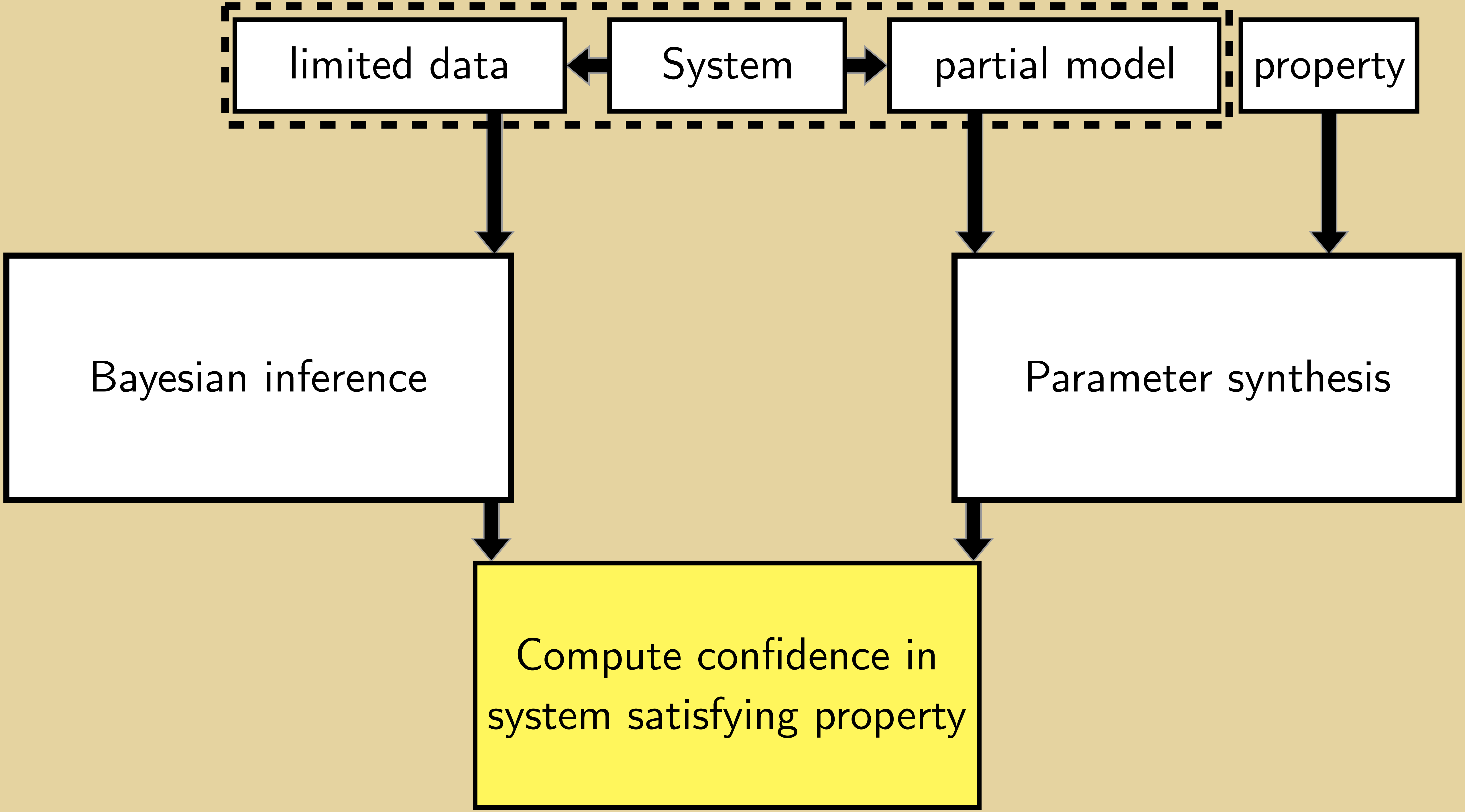

$$\frac{\text{COUNT}\,[1 - \theta_j] \quad 8}{\text{COUNT}\,[\theta_j] \quad 3}$$

$$p(\theta_j) = \mathrm{Dir}(D_{s_1} + D_{s_2} + \alpha_{s_1})$$




$$\frac{\text{COUNT}\,[1 - \theta_j] \quad 10}{\text{COUNT}\,[\theta_j] \quad 4}$$

# Overview



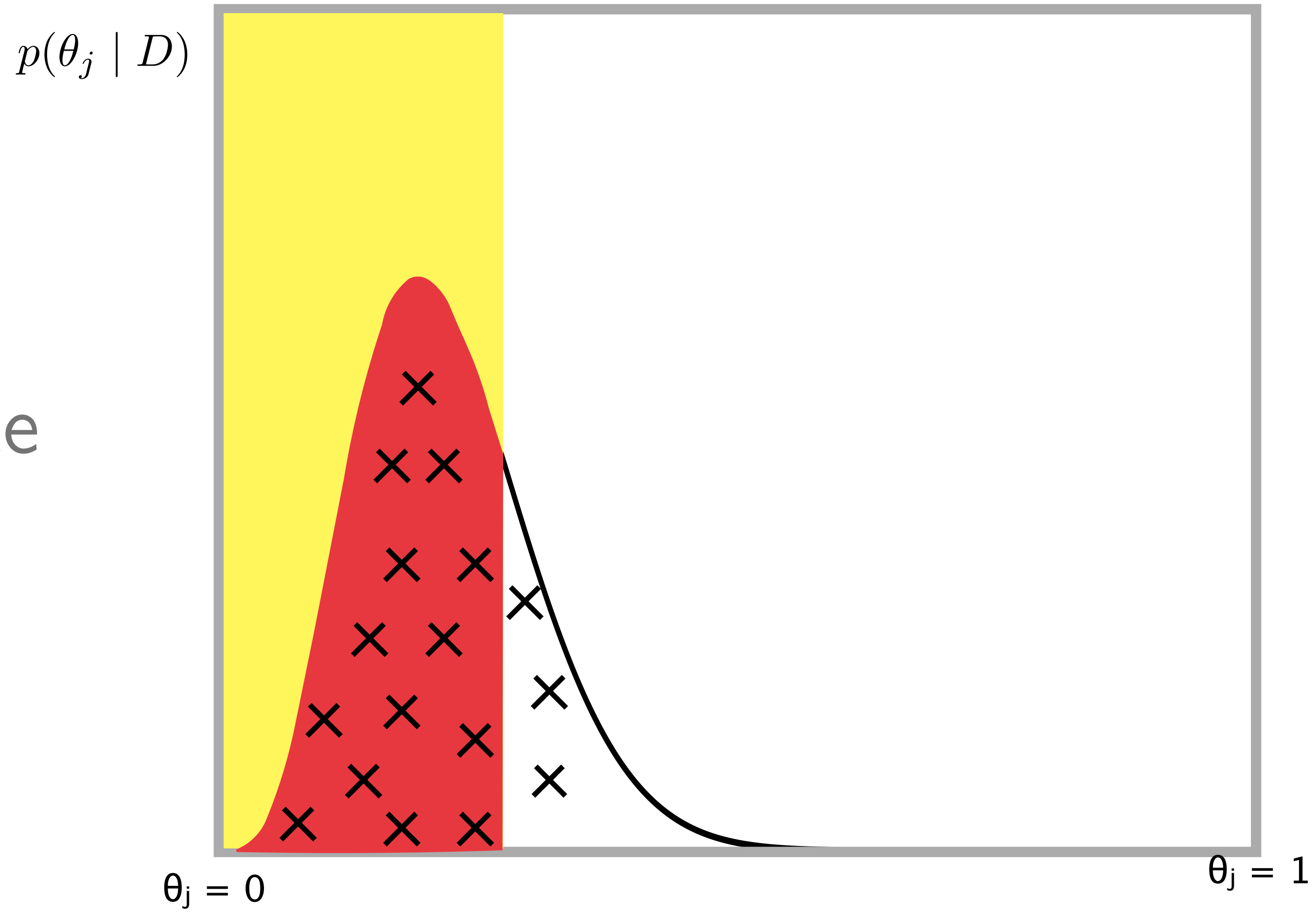
limited data
System
partial model
property
Bayesian inference
Parameter synthesis
Compute confidence in system satisfying property

# Confidence Calculation: basic pMC

$$\mathbb{P}(\mathbf{S} \models \phi \mid D) = \int_{\Theta_\phi} p(\theta \mid D) d\theta$$

$$\Theta_\phi = \{\theta \in \Theta : \mathbf{M}(\theta) \models \phi\}$$

Note: we use simple Monte Carlo to compute the integral.



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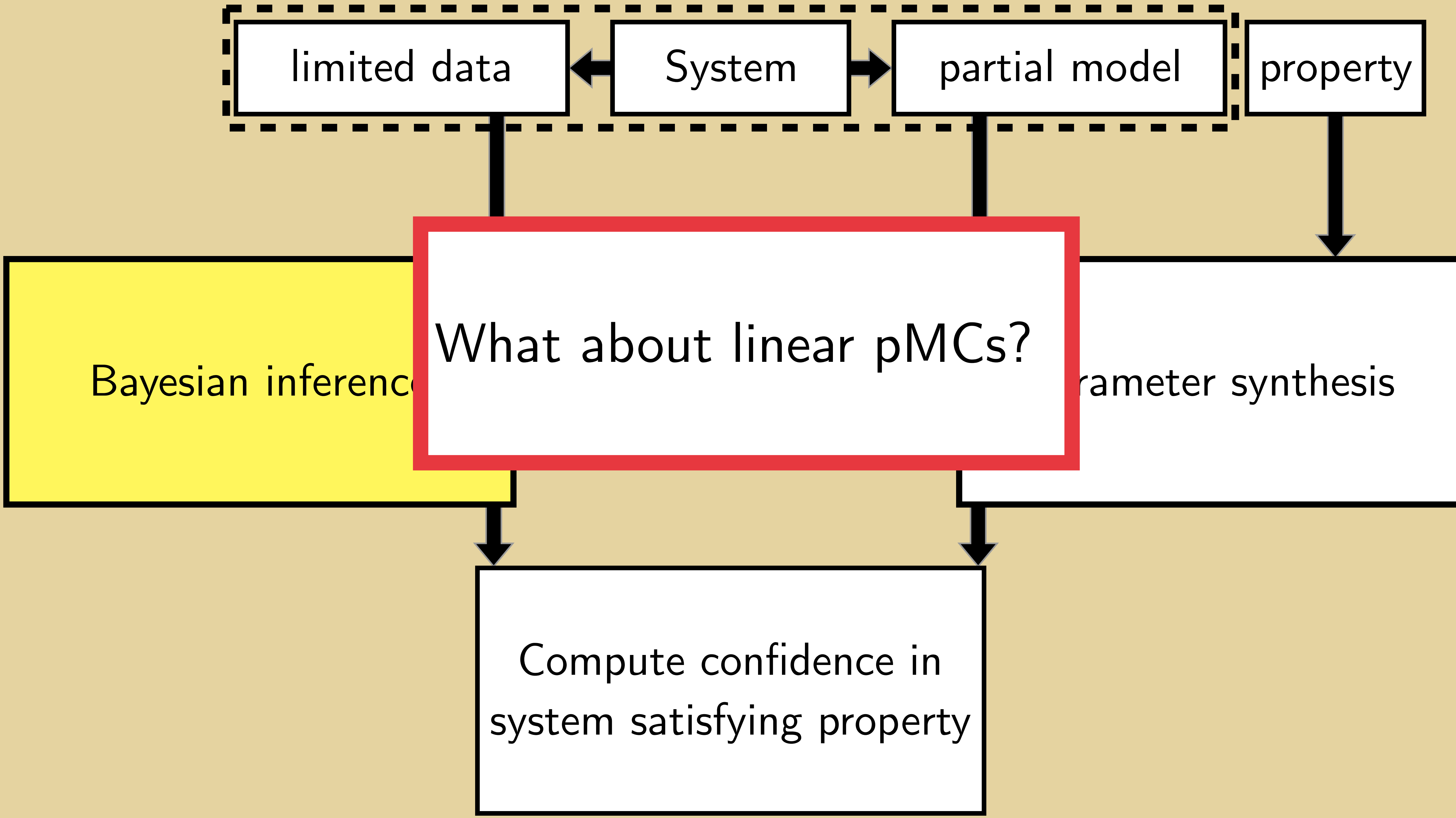
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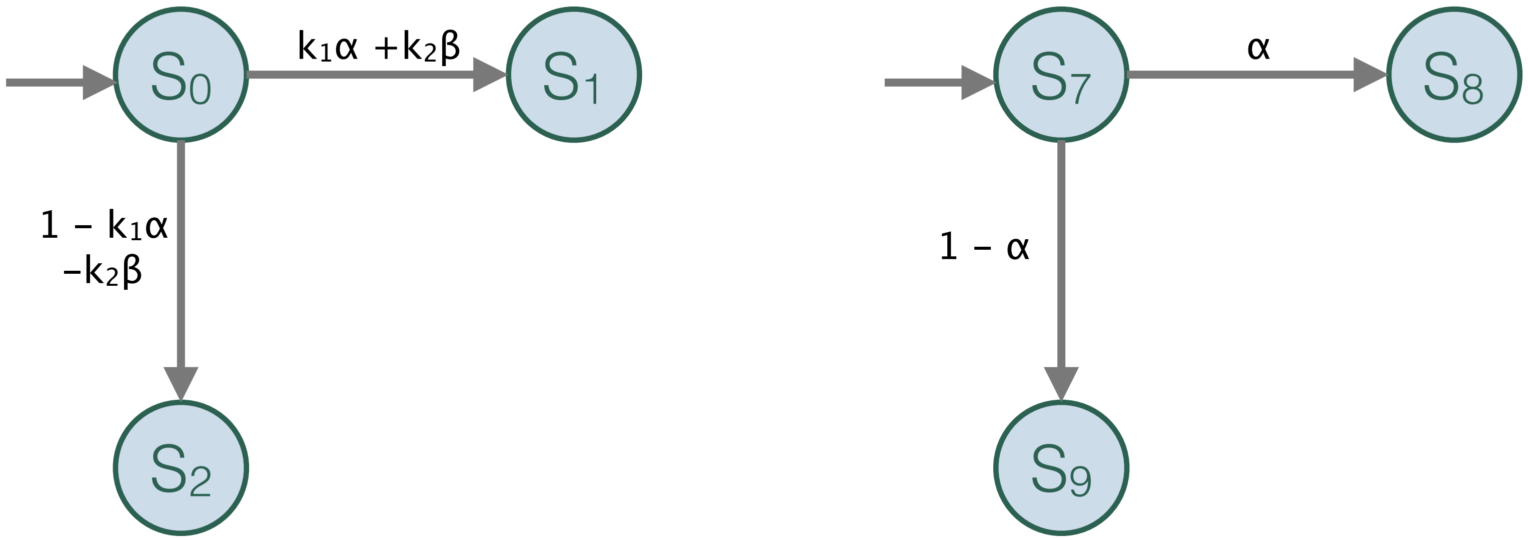
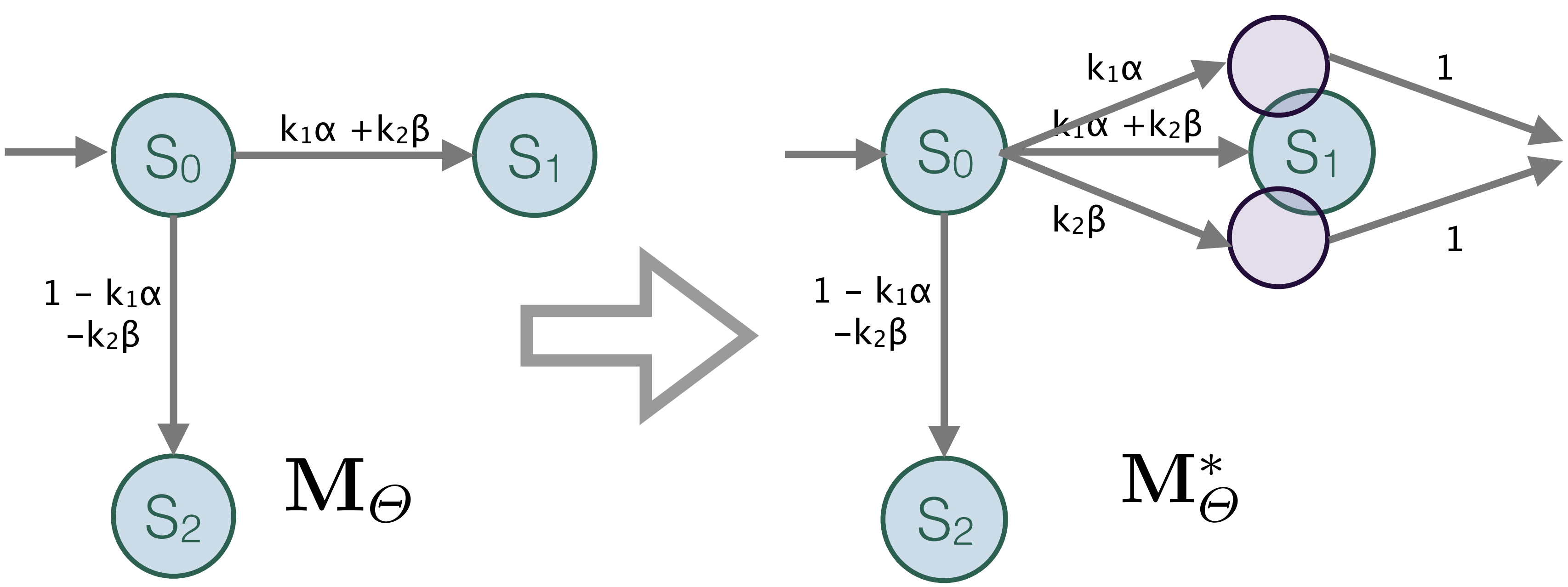
# Markov chain expansion

What if a parameter appears multiple times in a linear pMC, in different linear equations? How do we combine the posterior distributions?



# Markov chain expansion

We "expand" the transitions with linear parameterisation, to turn the MC into a basic pMC. i.e., transitions have only one parameter.



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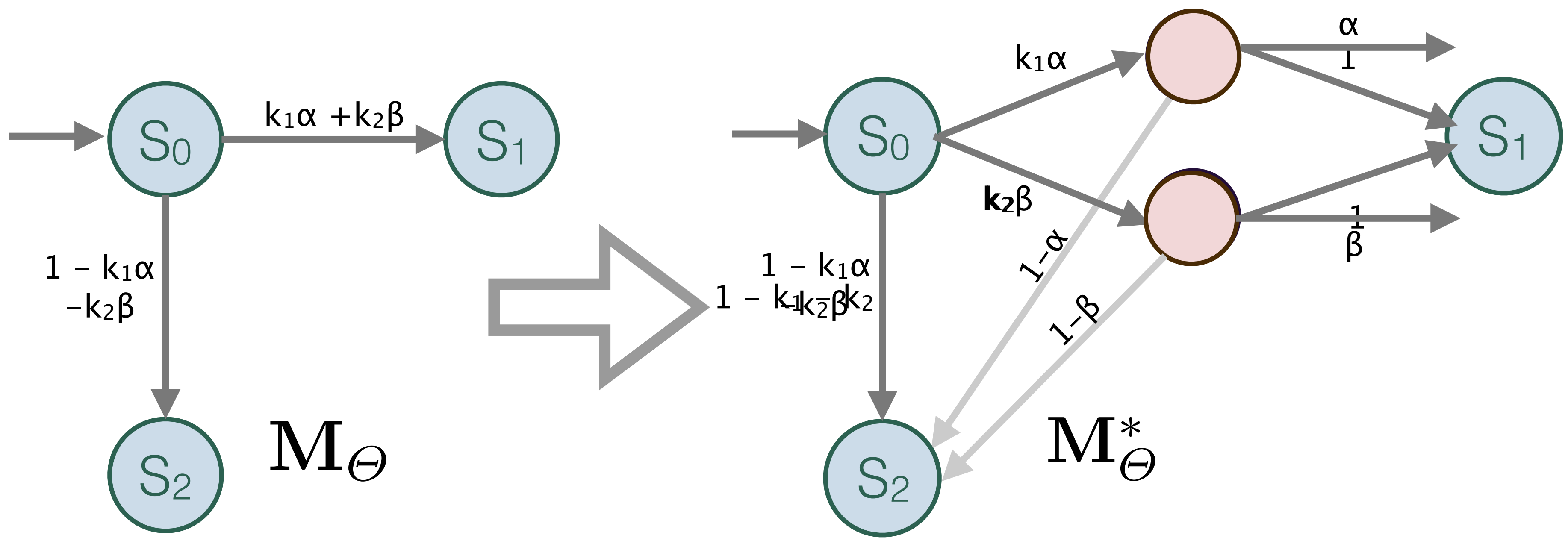
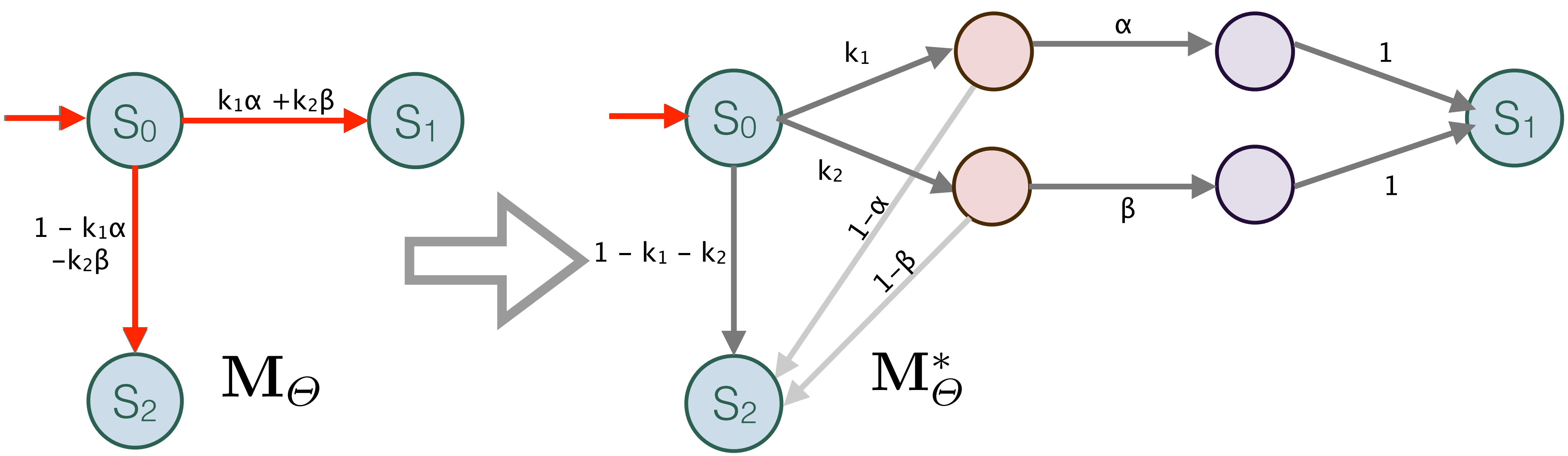


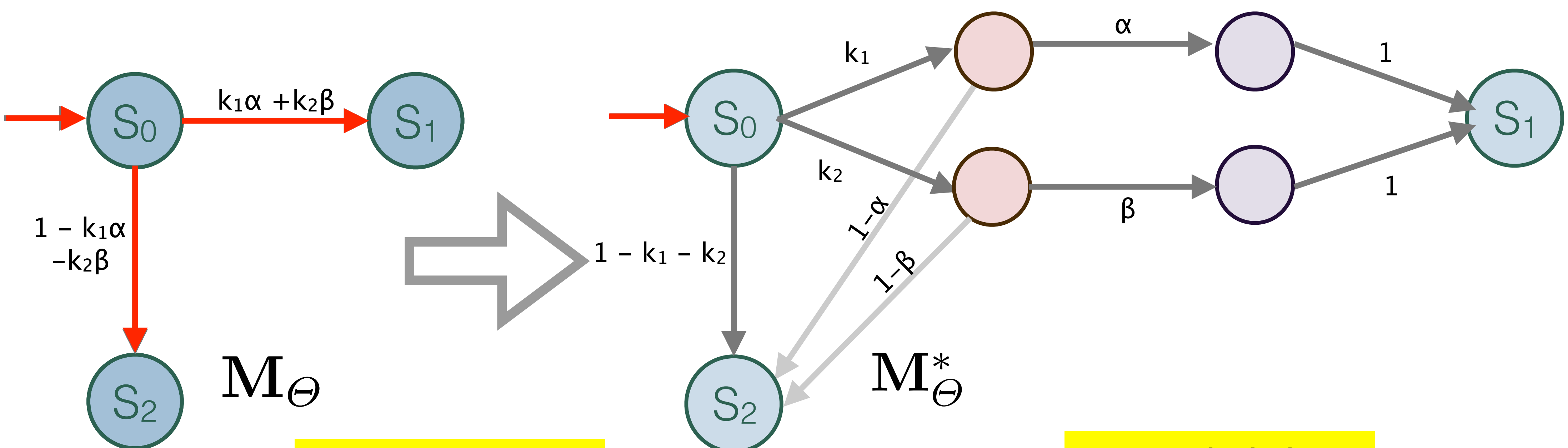
# Hidden Data

We now have a data set with gaps in. We know the transitions counts only for the original transitions.



# Hidden Data

We apply Bayes' rule




original data

expanded data

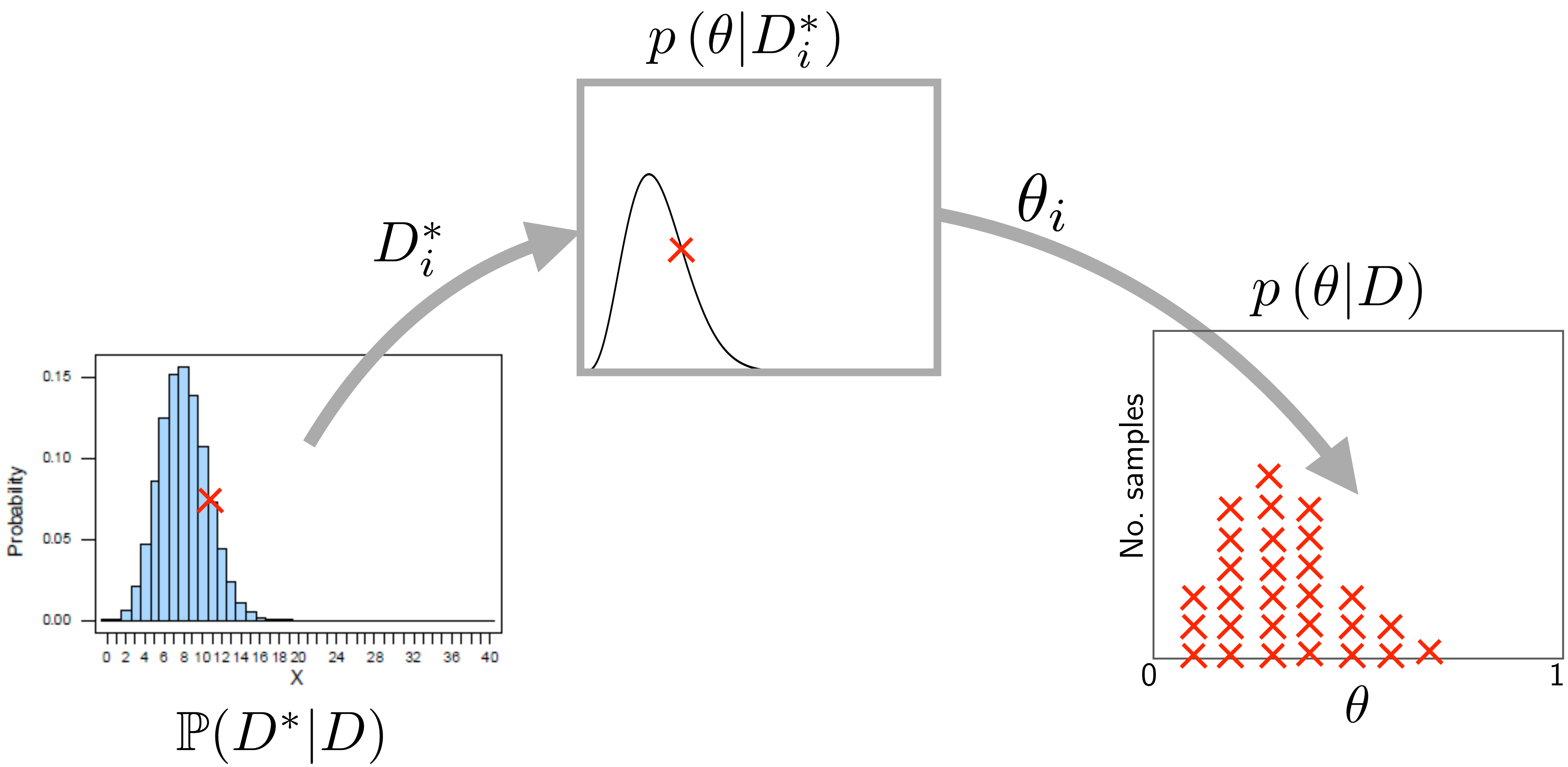
$$p\left(\theta|D\right) = \sum_{D^* \in \mathcal{D}^*} p\left(\theta|D^*\right)\mathbb{P}(D^*|D)$$

set of all possible completions of the expanded data

# Hidden Data

We use sampling to obtain a realisation of the posterior distribution, without evaluating the integral

$$p\left(\theta|D\right) = \sum_{D^* \in \mathcal{D}^*} p\left(\theta|D^*\right) \mathbb{P}(D^*|D)$$



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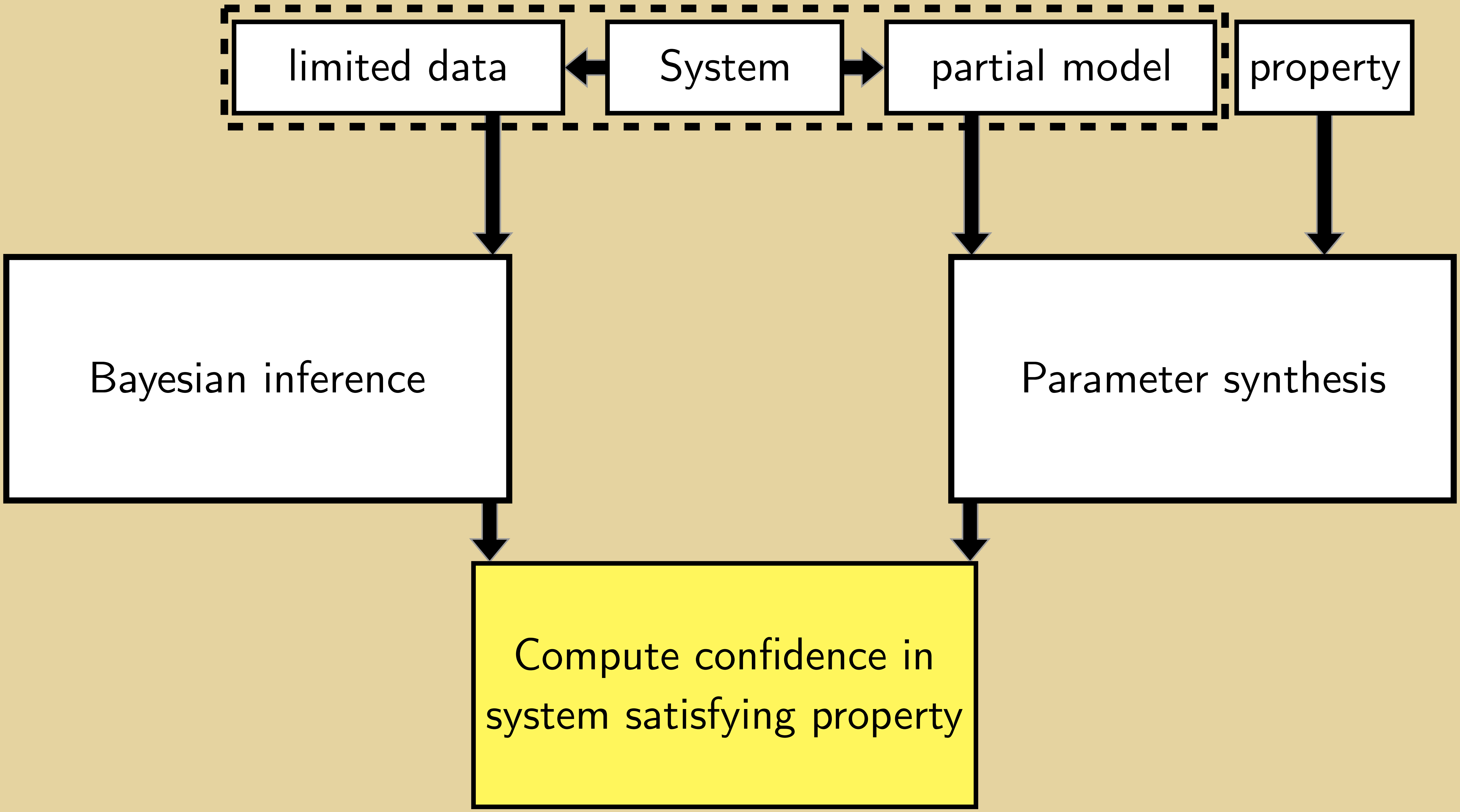

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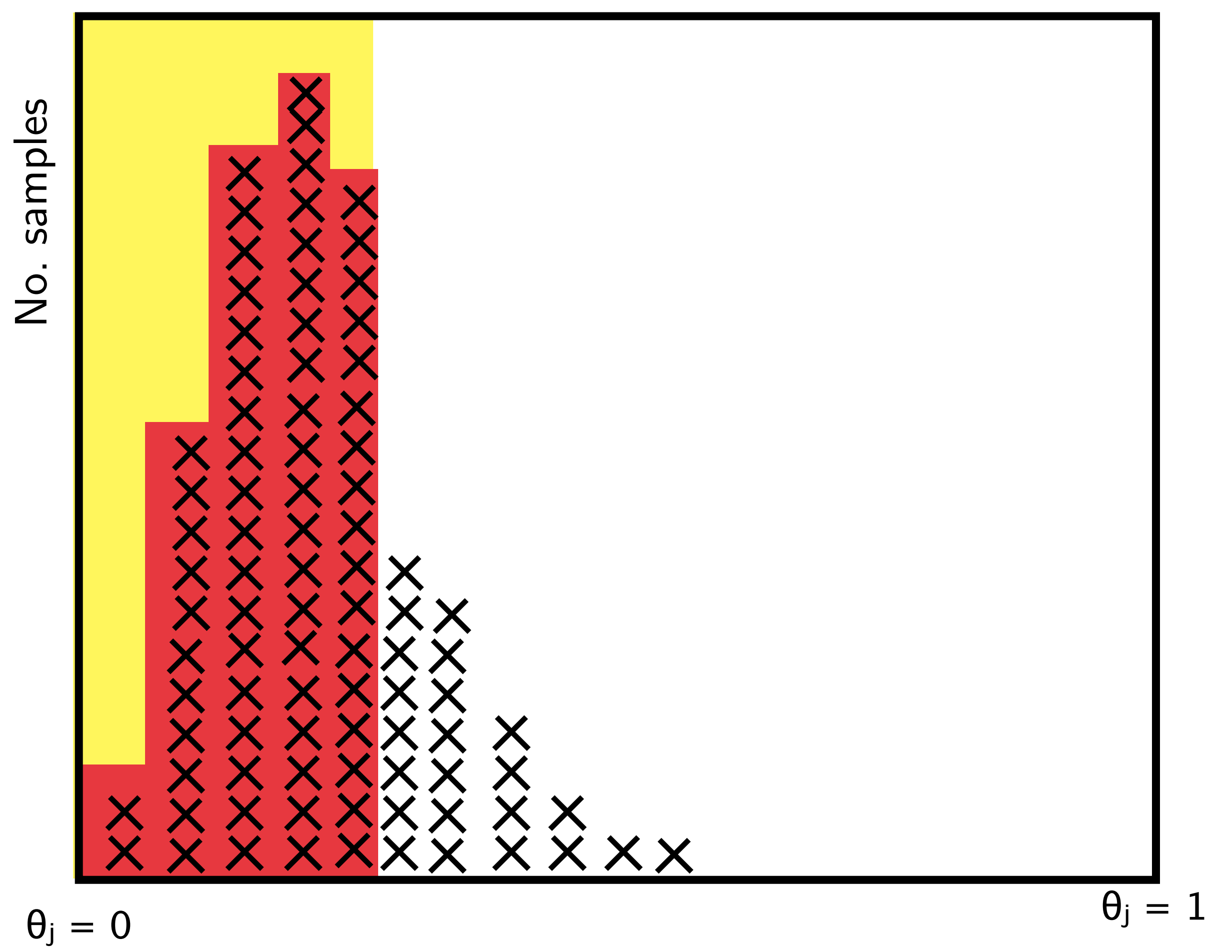
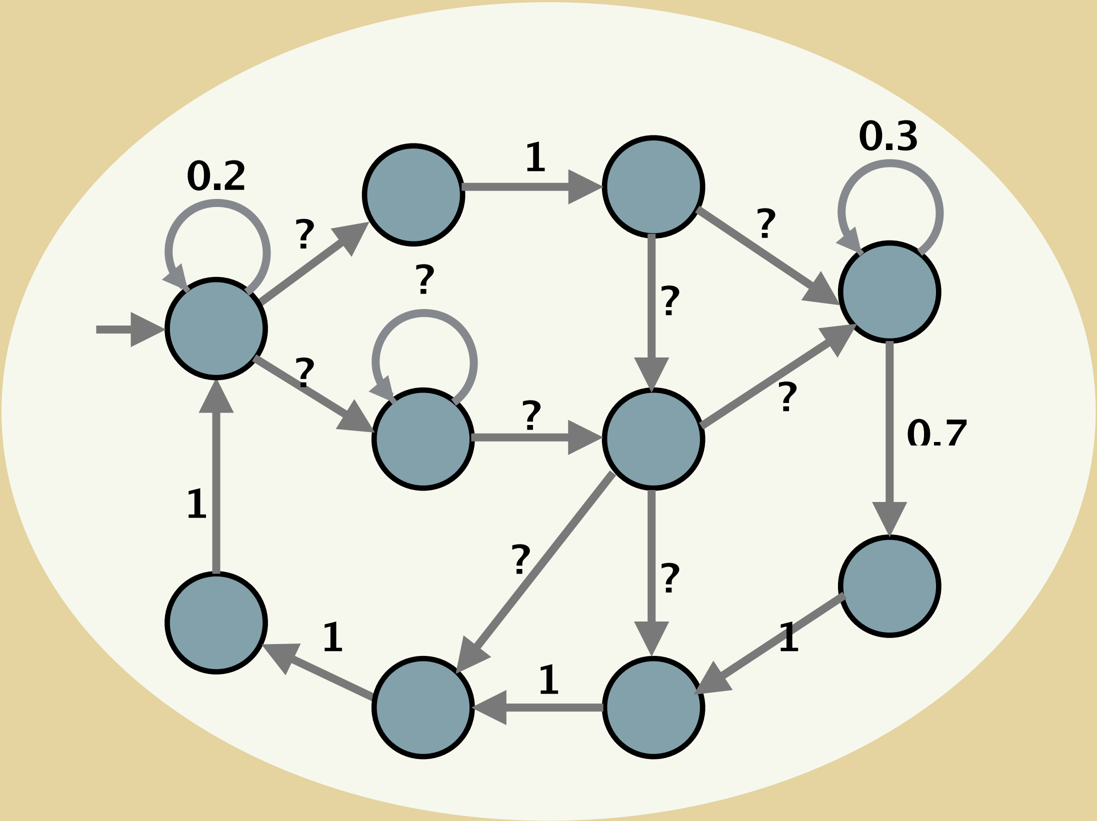


# Case Study

We run our approach over linear and basic parametric Markov chains, with a range of parameter values.

We implement a simple “black-box” statistical model checking algorithm for comparison



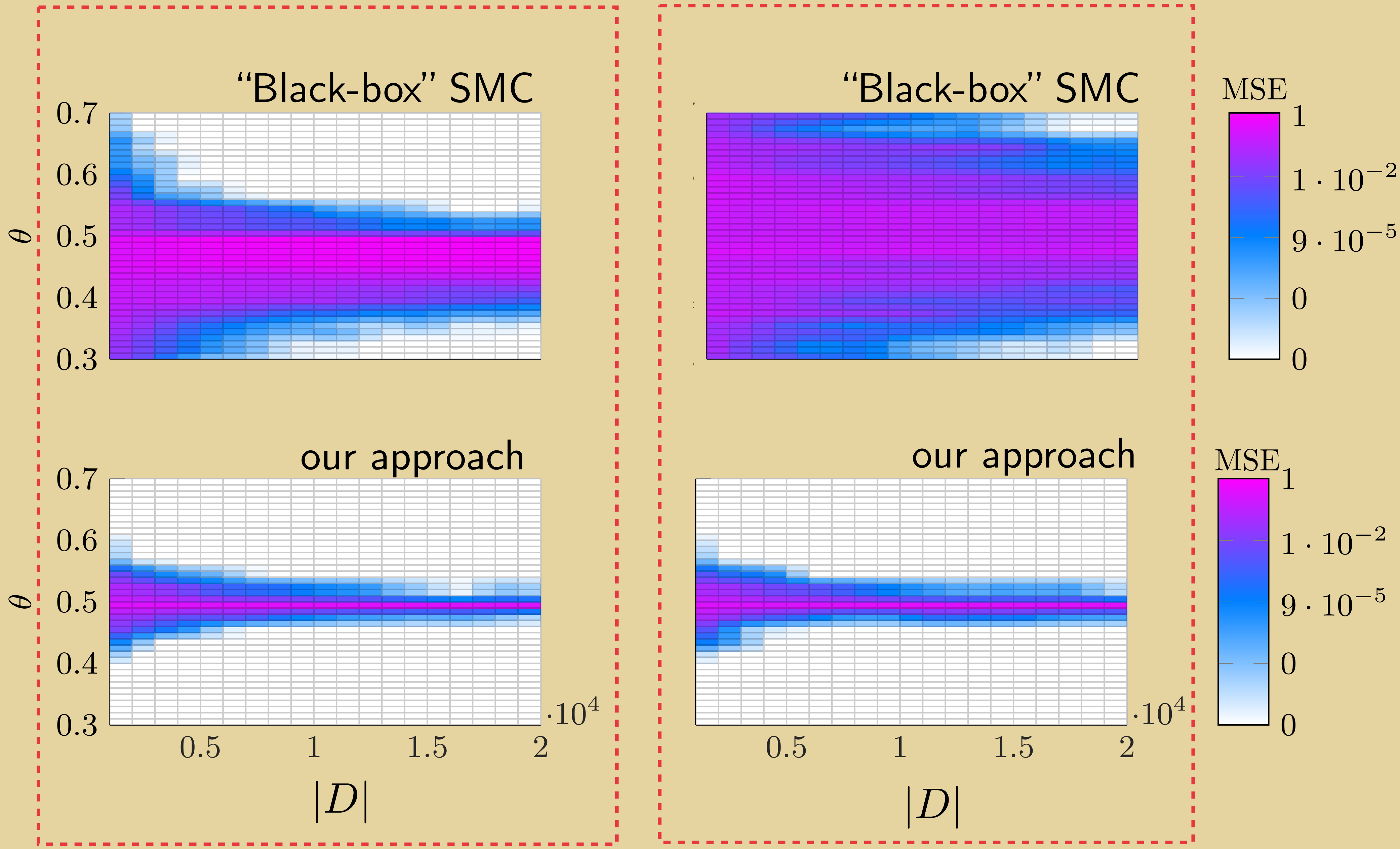
# Case Study

For our pMC and property: $\Theta_\phi = [0.5, 1]$

We compute a mean squared error (MSE):

$$MSE = \frac{1}{n} \sum_{i=1}^{\overset{\text{number of experiments}}{n}} (\underbrace{Y_{true}}_{Y_{true} = \begin{cases} 0 & \text{if } \theta \leq 0.5, \\ 1 & \text{if } \theta > 0.5, \end{cases}} - \overset{i\text{-th run } \mathbb{P}(\mathbf{M}_\theta \models \phi)}{Y_i})^2$$

# Case Study



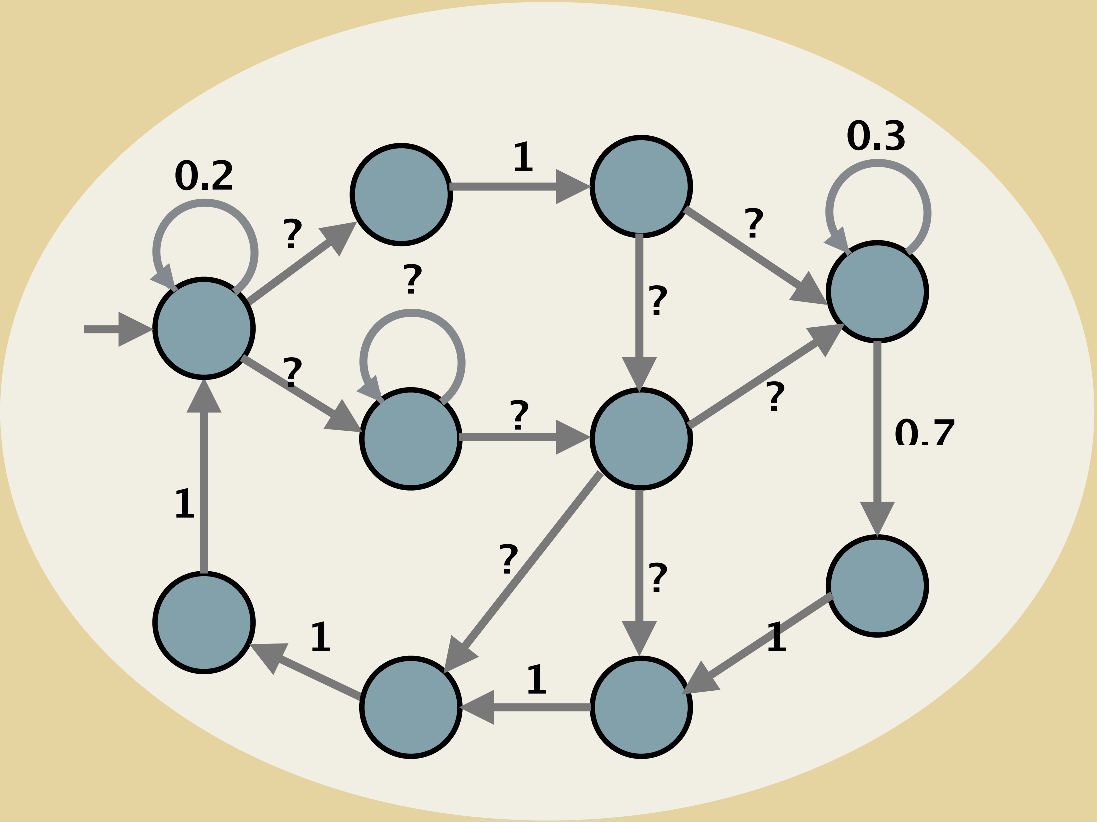

Data made up of traces of length 10

Data made up of traces of length 100

# Case Study

Our approach produces more accurate results with less data

We use information more efficiently: for SMC a “unit” of information is one trace; for us a “unit” of information is one parameterised transition



# Next steps

- Integration of alternative parameter synthesis techniques
- Non-linearly parameterised Markov chains
- External non-determinism (parameterised MDPs)
- Addition of Bayesian hypothesis testing

# Conclusions

- We presented a data-based verification approach
- Addresses model-checking with partial models and limited data
- Promises greater accuracy than black-box SMC

# References

1. This framework was originally proposed in *"Data-driven property verification of grey-box systems by Bayesian experiment design", S. Haesaert, P.M.J. Van den Hof, and A. Abate*

2. We use PRISM's parametric model checking tool: *"PARAM: a model checker for parametric Markov models", E.M. Hahn, H. Hermanns, B. Wachter, and L. Zhang*